PORTFOLIO GROWTH OPTIMIZATION WITH DOWNSIDE PROTECTION

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ABSTRACT

This study introduces growth optimization with downside protection as a portfolio selection technique based on the Power-Log utility function. The selected optimal portfolios extend from one that maximizes growth and is very risky, to portfolios with substantial downside protection. The technique can be used with all types of assets, including those with highly skewed return distributions and fat tails. This study uses a riskless asset, a stock-index and a call option to show that at every level of expected portfolio return, optimal Power-Log utility portfolios consistently have less downside risk and greater upside potential than the corresponding Markowitz mean-variance efficient portfolios.

Introduction

Harry Markowitz [8] laid out the mean-variance framework for portfolio construction that minimizes variance of return for a given expected return. Another approach to portfolio selection is based on multi-period portfolio theory, and draws on the work of Mossin [10], Hakansson [4] [5], Leland [7], Ross [11], Huberman and Ross [6], and Grauer and Hakansson [1] [2] [3]. This method typically uses the Savage [12], Von Neumann and Morgenstern [14] expected utility criterion for choice under uncertainty to select a portfolio, where the utility function is either a log function, or a power function. The log utility function is a special case of the power utility function, and it produces portfolios with maximum growth. The growth optimization with downside protection technique combines the maximum growth properties of the log utility function with the scalable downside protection features of the power utility function. It enables investors to construct portfolios with positively skewed returns that give them high upside potential along with downside protection. This study compares the portfolio compositions and return characteristics of Markowitz mean-variance efficient portfolios with portfolios constructed by using the growth optimization with downside protection technique. The portfolios are constructed from three assets: a riskless asset, a stock index, and an at the money call option on the stock index.

Methodology

The Power-Log utility function is a combination of a power function and the log function, and for that reason we refer to it as the Power-Log utility function. For returns less than zero the Power-Log utility function is a power function with power less than or equal to zero, and for returns greater than or equal to zero the Power-Log utility function is a log function.

$$\begin{aligned} U &= \frac{1}{\gamma}(1+r)^{\gamma} \quad \text{for } r < 0 \\ &= \ln(1+r) \quad \text{for } r \geq 0 \end{aligned} \tag{1}$$

where,

- r — portfolio return
- γ — power, is less than or equal to 0

The Power-Log utility function accommodates all investors. When the downside power is zero, the Power-Log utility function is the same as the log utility function, which maximizes portfolio growth over time. Maximizing portfolio growth is a natural goal for all investors, but typically involves more risk than most investors are willing to accept. An investor who is willing to take that risk can select the maximum growth portfolio by using a downside power function with a power of zero. An investor who wants more downside protection can use a downside power function with a lower value for the power. Lower powers provide greater downside protection. Another interesting characteristic of the Power-Log utility function is that it is continuously differentiable. The slope of the log function is 1 when the return is zero, and the slope for all the power functions is also 1 when the return is zero. The Power-Log function does not have a kink at a return value of zero. This feature allows the use of fast mathematical programming algorithms for portfolio optimization.

An optimal portfolio is constructed by selecting asset weights that maximize the expected utility of the portfolio for a given Power-Log utility function, using the joint return distribution of asset returns as an input to the optimization algorithm. We simulate the joint return distribution for the riskless asset, the stock index, and the at the money call option on the stock index. The joint return distribution contains 10,000 observations. The assumed one-period returns for the assets correspond to annual returns observed in U.S. capital markets. The riskless asset is assumed to have a return of 4%. The log of the stock index return is assumed to have a mean of 10% and a standard deviation of 20%. Assuming that the call has one year to expiration and is held to expiration, and that the stock index pays no dividends, the call price is calculated by using the Black-Scholes option pricing model for European options. The call's price is in turn used to simulate its return distribution contingent on the simulated expiration date stock index prices. The call's return distribution has a mean of 64.54%, a standard deviation of 188.48%, and a skewness of 1.39. The correlation between the call return and the stock return is 0.96. To make it easier to compare and interpret the results, all the optimal portfolios have been constructed with a no short sales constraint on the stock index and the call option.

Results

Table I shows the portfolios constructed by using the Power-Log utility function for downside powers of 0 to –50 in the Optimal Power-Log Portfolios panel, and their expected returns are shown in the second column of the table. The corresponding Markowitz mean-variance (M-V) efficient portfolios shown in the table have been constructed to match the expected return of the optimal Power-Log portfolios.

Table I

		------------ Optimal Power-Log Portfolios ------------				------- M-V Efficient Portfolios -------		
Portfolio	Expected Return (%)	Downside Power	Riskless Weight (%)	Stock Weight (%)	Call Weight (%)	Riskless Weight (%)	Stock Weight (%)	Call Weight (%)
1	21.93	0.00	-58.41	150.62	7.79	-104.97	204.97	0.00
2	17.85	-0.90	1.89	87.99	10.11	-58.36	158.36	0.00
3	15.78	-1.70	27.46	62.11	10.43	-34.59	134.59	0.00
4	12.66	-4.00	58.49	31.84	9.67	1.01	98.99	0.00
5	7.25	-50.00	93.31	1.56	5.13	62.85	37.15	0.00

The optimal Power-Log portfolio in Row 1 of Table I has been constructed with a downside power of 0 and is the maximum growth portfolio, since the Power-Log utility function with a downside power of 0 is the log utility function. It is a very risky, leveraged portfolio with a short position in the riskless asset of 58.41%, but is expected to have the highest growth rate over time. As the downside power is lowered, more downside protection is built into the portfolios and the resulting optimal Power-Log portfolios are progressively more conservative. Remarkably, the weight of the call option relative to the weight of the stock in these portfolios increases as portfolio risk declines. This is due to the significant positive skewness in the call option's returns.

The M-V efficient portfolios shown in Table I have the lowest variance for the given expected returns. All of them have a negligible position in the call option. Clearly the positive skewness of the call's returns carries no weight in the construction of M-V efficient portfolios. The leveraged stock portfolios dominate the call option in mean-variance space. Even though the 0.96 correlation between the stock and call returns is less than 1, the resulting percent weight of the call option in all the M-V efficient portfolios is zero to two decimal places. The M-V efficient portfolio in Row 1 uses a great deal of direct leverage, with a short position in the riskless asset of 104.97%, to achieve an expected return of 21.93%. For lower levels of expected return, the amount of leverage in the M-V efficient portfolio declines, until it is approximately zero in Row 4. The corresponding Power-Log portfolio in Row 4 is very different; it has a large weight of 58.49% in the riskless asset, a smaller weight of 31.84% in the stock and a significant weight of 9.67% in the call option. The return characteristics of these two portfolios, as well as the other pairs of optimal Power-Log and M-V efficient portfolios are very different as well.

Table II compares the minimum, maximum, standard deviation and negative semideviation below zero for the return distributions of the optimal portfolios. The entire range of returns for each optimal Power-Log portfolio lies above that for the corresponding M-V efficient portfolio, showing that the Power-Log portfolios consistently provide better downside protection and greater upside potential than M-V efficient portfolios. Interestingly, for expected returns above 20%, all the mean-variance efficient portfolios have a minimum return of "–100%," i.e., bankruptcy. Markowitz mean-variance efficiency can be fatal for high-risk high-return portfolios!

Table II

Portfolio	Expected Return (%)	Minimum Return (%) Power-Log	Minimum Return (%) M-V Efficient	Maximum Return (%) Power-Log	Maximum Return (%) M-V Efficient	Standard Deviation (%) Power-Log	Standard Deviation (%) M-V Efficient	Neg. Semidev. (%) Power-Log	Neg. Semidev. (%) M-V Efficient
1	21.93	-91.11	-100.00	310.67	282.84	48.54	46.67	17.03	18.38
2	17.85	-57.35	-87.47	255.89	219.42	38.69	36.06	12.15	13.78
3	15.78	-42.72	-73.75	224.77	187.10	33.45	30.65	9.81	11.43
4	12.66	-24.45	-53.18	173.67	138.66	25.25	22.54	6.46	7.93
5	7.25	-2.23	-17.46	73.15	54.53	10.00	8.46	0.90	1.98

The standard deviation for the M-V efficient portfolios is slightly lower than the standard deviation for the corresponding Power-Log portfolios, since the mean-variance efficient portfolios have been constructed to minimize the standard deviation for a given expected return. However, standard deviation is an inappropriate measure of risk for assets with asymmetric return distributions. The Power-Log portfolios' larger positive deviations of return from their expected value, when compared to the mean-variance efficient portfolios, appear incorrectly as larger contributions to risk. These larger positive deviations are in fact highly desirable, and should not be penalized. Negative semideviation is a better measure of risk. It focuses on downside returns only, and is a measure of downside risk (Markowitz [9]). The negative semideviation for all the Power-Log portfolios is smaller than the negative semideviation for the corresponding M-V efficient portfolios. Based on this measure, the Power-Log portfolios have consistently lower risk than the M-V efficient portfolios.

Table III compares the asymmetry characteristics of the return distributions of the optimal portfolios. Predictably, all the M-V efficient portfolios have exactly the same skewness as the stock return distribution, 0.61, since they are all leveraged stock portfolios with negligible call option positions, while the skewness of the Power-Log portfolios increases consistently as risk decreases. The results clearly show the benefits of the Power-Log portfolios relative to the M-V efficient portfolios for investors who have a preference for positively skewed returns.

Table III

		Skewness		Value at Risk (%)		Value to Gain (%)		VtG / VaR	
Portfolio	Expected Return (%)	Power-Log	M-V Efficient	Power-Log	M-V Efficient	Power-Log	M-V Efficient	Power-Log	M-V Efficient
1	21.93	0.90	0.61	55.71	66.24	161.08	148.99	2.89	2.25
2	17.85	1.06	0.61	36.67	50.26	131.88	116.01	3.60	2.31
3	15.78	1.13	0.61	28.13	42.12	115.61	99.20	4.11	2.36
4	12.66	1.23	0.61	16.97	29.92	89.27	74.02	5.26	2.47
5	7.25	1.37	0.61	1.87	8.73	38.39	30.28	20.55	3.47

With the widespread use of derivatives and the need to control the risk exposure of portfolios and financial institutions, Value at Risk, or VaR, has gained popularity as a measure of exposure to large losses (Smithson [13]). It focuses on the probability and size of losses in the downside tail of the return distribution. At a 99% confidence level, if VaR is $5 million then there is a 1% probability of a loss of $5 million or higher. Since the Power-Log portfolios have asymmetric return distributions with fatter tails on the upside and thinner tails on the downside, a complementary measure for the potential value being added on the upside is the "Value to Gain," or VtG. At a 99% confidence level if VtG is $50 million, then there is a 1% probability of a gain of $50 million or higher. The VaR and VtG are measures of location for the tails of the distribution and can be used for dollar, or percent losses and gains.

Table III shows the VaR and VtG for the optimal portfolios at the 99% confidence level. The VaR for the Power-Log portfolios is consistently lower at all levels of expected return than the VaR for the corresponding M-V efficient portfolios, and the difference in VaR is quite dramatic for the most conservative portfolios. The Power-Log portfolios do a better job of protecting investors against a large loss. The VtG for the Power-Log portfolios is also consistently higher than the VtG for the M-V efficient portfolios. Besides doing the better job in protecting the downside, the Power-Log portfolios also do a better job of delivering higher potential gains to the investor.

The ratio of VtG to VaR gives us an idea of the asymmetry in the tails of the portfolio return distribution with regard to gains and losses. It is different from skewness, since the skewness measure does not focus on the tails of the distribution exclusively, as does the VtG to VaR ratio. The last panel of Table III shows the VtG to VaR ratios for the optimal portfolios at the 99% confidence level. At a given level of expected return, the ratio shows the size of the potential gains relative to the size of the potential losses at the extremes. As expected, the Power-Log portfolios dominate at all levels of expected return, and much more so for the conservative portfolios. As we move from risky portfolios to conservative portfolios the M-V technique pulls in the tails of the distribution symmetrically, resulting in a relatively flat VtG/VaR ratio, while the Power-Log technique pulls in the downside tail of the distribution much more than it pulls in the upside tail, resulting in dramatically higher VtG/VaR ratios for the conservative portfolios.

Conclusion

This study introduces a portfolio selection technique based on the Power-Log utility function. It uses a riskless asset, a stock index and a call option to construct optimal portfolios, and show that the portfolios constructed by using the Power-Log utility function are consistently superior to Markowitz mean-variance efficient portfolios for investors who want to maximize portfolio growth and for investors who are interested in reducing the downside risk in their portfolios without giving up too much upside potential. Portfolio selection based on the Power-Log utility function will be particularly useful in the management of hedge funds, bond portfolios, bank loan portfolios and insurance portfolios, since the assets in these portfolios have asymmetric returns.

REFERENCES

[1] Grauer, R. R., & Hakansson N. H. Higher Return, Lower Risk: Historical Returns on Long-Run, Actively Managed Portfolios of Stocks, Bonds and Bills, 1936-1978. *Financial Analysts Journal*, 1982, 38, 39-53.

[2] Grauer, R. R., & Hakansson N. H. Returns on Levered, Actively Managed Long-run Portfolios of Stocks, Bonds and Bills, 1934-1983. *Financial Analysts Journal*, 1985, 41, 24-43.

[3] Grauer, R. R., & Hakansson N. H. On the Use of Mean-variance and Quadratic Approximations in Implementing Dynamic Investment Strategies: A Comparison of Returns and Investment Policies. *Management Science*, 1993, 39 (7), 856-871.

[4] Hakansson, Nils H. On Optimal Myopic Portfolio Policies, With and Without Serial Correlation of Yields. *Journal of Business*, 1971, 44, 324-334.

[5] Hakansson, Nils H. Convergence to Isoelastic Utility and Policy in Multiperiod Portfolio Choice. *Journal of Financial Economics*, 1974, 1, 201-224.

[6] Huberman, G., & Ross S. Portfolio Turnpike Theorems, Risk Aversion and Regularly Varying Utility Functions. *Econometrica*, 1980, 51, 1345-1361.

[7] Leland, H. On Turnpike Portfolios. *Mathematical Methods in Investment and Finance.* Karl S. & Szego, G. P. (eds.). Amsterdam, North-Holland, 1972.

[8] Markowitz, H. M. Portfolio Selection. *Journal of Finance*, 1952, 7 (1), 77-91.

[9] Markowitz, H. M. *Portfolio Selection – Efficient Diversification of Investments*. NewYork, John Wiley & Sons, 1959.

[10] Mossin, J. Optimal Multiperiod Portfolio Policies. *Journal of Business*, 1968, 41, 215-229.

[11] Ross, S. Portfolio Turnpike Theorems for Constant Policies. *Journal of Financial Economics*, 1974, 1, 171-198.

[12] Savage, L. *The Foundations of Statistics*. New York, Wiley, 1964.

[13] Smithson, C. W. *Managing Financial Risk*. New York, Irwin, 1998.

[14] Von Neumann, J. & Morgenstern, O. *Theory of Games and Economic Behavior*. Princeton, Princeton University Press, 1953.