OPTIMAL HEDGING FOR DOWNSIDE PROTECTION

Jivendra K. Kale, Graduate Business Programs, Saint Mary's College of California, 1928 Saint Mary's Road, Moraga, CA 94556. (925) 631-4178, jkale@stmarys-ca.edu

ABSTRACT

In this study we compare portfolios constructed by using a portfolio insurance strategy implemented with European put options with optimally hedged portfolios constructed by using an expected utility maximization with Power-Log utility functions. We use a riskless asset, a stock index and put options on the stock index to construct portfolios. We find that the optimally hedged portfolios have return distributions with risk and asymmetry characteristics that are consistently superior to those for the return distributions of the corresponding portfolio insurance portfolios with the same expected returns.

INTRODUCTION

Downside protection for investment portfolios is typically implemented as a portfolio insurance strategy and seeks to limit losses, such that a portfolio's value is either equal to or exceeds a prespecified floor value on the horizon date. It can be implemented with put options and dynamic hedging techniques. The dynamic hedging techniques include the option based technique developed by Rubinstein and Leland [32], and constant proportion portfolio insurance developed by Perold [24], described by Perold and Sharpe [25] for fixed income portfolios, and also described by Black and Jones [2] for equity portfolios. The option based dynamic hedging technique dynamically allocates the portfolio to positions in the risky asset and the riskless asset to replicate a put option, while the constant proportion portfolio insurance method dynamically allocates an amount equal to a prespecified multiple times the excess of the portfolio value over the floor, to the risky asset.

Portfolio insurance was widely used and publicized as a portfolio management strategy before the market crash of October 1987. In theory it guarantees a floor value for the portfolio, but real world conditions including jumps in security prices, basis risk from horizons that do not match the maturities of securities available in the market as well as less than perfect correlation between market indexes and managed portfolios, and other factors listed by Rubinstein [29] add uncertainty to the final outcome. In practice there is typically no guarantee that the value of the portfolio will not drop below the floor value. This was amply demonstrated by the failure of portfolio insurance strategies during the market crash of October 1987.

The traditional goal of downside protection can be described as deterministic, in that a fixed floor is specified for the portfolio value. Given the uncertainty associated with achieving this goal in the real world, a probabilistic approach might be more appropriate. For example, to compare the downside exposure of two portfolios we can use Value at Risk (VaR), which is a probabilistic measure. VaR has gained popularity as a measure of exposure to losses (Smithson [35]). At a 95% confidence level, if VaR is $1 million, then there is a 5% probability of a loss of $1 million or more. A portfolio with a lower value of VaR is likely to have smaller losses than a portfolio with a higher value of VaR. A probabilistic approach also allows us to use portfolio construction methods that are not based on a deterministic floor value for the portfolio. These methods have the potential for constructing portfolios that have characteristics that are superior to those produced by traditional portfolio insurance techniques, and at the same time achieve the investor's goal for controlling downside exposure.

In this study we compare portfolios constructed by using option based portfolio insurance implemented with European put options with optimally hedged portfolios constructed by using expected utility maximization (Savage [34], and Von Neumann and Morgenstern [36]) with the family of Power-Log utility functions described in Kale [12]. We selected the Power-Log utility function technique to construct the optimal hedges, since it is very effective in accounting for the asymmetry in the option return distributions as well as the different levels of loss aversion for different investors. Portfolio selection with Power-Log utility functions draws on multiperiod portfolio theory, which has been discussed extensively in the literature by Kelly [13], Markowitz [18, 19, 20], Latane [13], Brieman [3, 4], Pratt [26], Latane and Tuttle [15], Mossin [23], Samuelson [31, 32, 33], Merton [21], Hakansson [8, 9, 10], Leland [16], Ross [28], Merton and Samuelson [22], Huberman and Ross [11], Grauer and Hakansson [5, 6, 7], MacLean and Ziemba [17] and others.

Power-Log utility functions combine the maximum growth properties of the log utility function with the scalable downside protection features of the power utility function, and are particularly suited to the construction of portfolios with low downside exposure. Rendleman and McEnally [27] compared portfolio insurance to expected utility maximization using a log utility function for portfolio construction, but their study was limited since the log utility function does not have the scalable downside protection features of Power-Log utility functions.

PORTFOLIO CONSTRUCTION METHODOLOGIES

An investor can use the option based portfolio insurance technique to limit downside exposure by selecting a floor value for the portfolio, and then buying a put option on the portfolio with a strike price that corresponds to the selected floor value. While the market crash of October 1987 appears to have discredited option based portfolio insurance, there is plenty of evidence that investors still use put options on stock indexes to protect their portfolios. The December 2, 2004 issue of the Wall Street Journal reported an open interest of 2,566,101 put option contracts on the S&P500 index on the previous day, which was approximately twice the open interest in call options on the index on that day. The S&P500 index closed at 1191.37, and trading volume exceeded 1,000 contracts on out of the money put options on the index for strike prices ranging from 1190 down to 950, which is approximately 80 percent of the value of the index. This data indicates that investors are actively using option based downside protection for protecting their portfolio with floor values that go down to 80 percent of portfolio value. The selection of the floor value for a portfolio typically depends on the investor's aversion to losses, and the range of strike prices at which these put options are trading suggests that investors' aversion to losses varies widely.

Optimal hedging is an alternative to setting a deterministic floor value for the portfolio. Portfolio selection with Power-Log utility functions allows investors to select a portfolio that reflects their loss aversion. A Power-Log utility function is a combination of a power function and the log function. For losses the Power-Log utility function is a power function with power less than or equal to zero, and for gains the Power-Log utility function is a log function.

$$\begin{aligned} U &= \frac{1}{\gamma}(1+r)^{\gamma} \quad \text{for } r < 0 \\ &= \ln(1+r) \quad \text{for } r \geq 0 \end{aligned} \tag{1}$$

where,

- r   portfolio return
- γ power, is less than or equal to 0

The family of Power-Log utility functions accommodates all investors. When the downside power is zero, the Power-Log utility function is the same as the log utility function, which results in the selection of maximum growth portfolios (Kelly [13]). While maximizing portfolio growth is a natural goal for all investors, typically the selected portfolios are much more risky than most investors want. An investor who is willing to accept a large amount of risk can select the maximum growth portfolio by using a Power-Log utility function with a downside power of zero, while an investor who wants less downside exposure can use a Power-Log utility function with a lower value for the downside power.

An interesting and desirable characteristic of Power-Log utility functions is that they are continuously differentiable. The slope of the log function is 1 when the return is zero, and the slope for all power functions is also 1 when the return is zero. As a result, Power-Log functions do not have a kink at a return value of zero. This feature allows the use of fast mathematical programming algorithms for portfolio optimization. The algorithm used for this study is a nonlinear mathematical programming algorithm based on an accelerated conjugate direction method developed by Best and Ritter [1], and has a superlinear rate of convergence.

To compare portfolios produced by portfolio insurance with optimally hedged portfolios selected by using Power-Log utility functions we use a time horizon of one year. The portfolio insurance portfolios are constructed by using a stock index and put options on the stock index with appropriate strike prices. The optimally hedged portfolios are constructed from the stock index, an at-the-money (ATM) put option on the stock index and a riskless asset. Any put option or combination of put options and call options could have been used for constructing the optimally hedged portfolios, but we decided use only the ATM put option for the sake of simplicity. The assumed one-period returns for these assets correspond to annual returns that have been observed in U.S. capital markets. The riskless asset is assumed to have a return of 4%. The stock index is assumed to have a current price of $100, and the log of the stock index return is assumed to have a mean of 10% and a standard deviation of 20%. Assuming that the put options have one year to expiration and are held to expiration, and that the stock index pays no dividends, we use the Black-Scholes option pricing model for European call options and put-call parity to calculate put prices.

To construct the optimally hedged portfolios with Power-Log utility functions, we simulate the joint return distribution of asset returns and use that as input to the optimization algorithm. We start by simulating the stock index returns for the one-year horizon, and use those with the beginning of year stock index price of $100 to generate end of year stock prices. We then calculate the corresponding expiration values of the put options, and use those with the beginning of year put prices to generate put option returns. The resulting joint return distribution for the riskless asset, stock index and put options contains 10,000 observations.

RESULTS

For $100 invested in the stock index we construct portfolio insurance portfolios with end of year floor values of $80, $85, $90, and $95. These floor values for the portfolios are the strike prices of the put options that are purchased to insure each portfolio respectively, and also represent the floor values as a percent of the investment in the stock index. They correspond to the range of actively traded put options on the S&P500 index reported in the December 2, 2004 issue of the Wall Street Journal. The Black-Scholes put option prices for the four strike prices are $0.77, $ 1.47, $2.53 and $4.03 respectively, and the price of the ATM put option with a strike price of $100 is $6.00.

We calculate the expected return for each portfolio insurance portfolio by using the simulated joint return distribution. Table I shows the four portfolios, their floor values as a percent of the amount invested in the stock index, their expected returns, and the investment weights of the stock index and corresponding put option in each portfolio insurance portfolio. The expected return for the portfolio insurance portfolios decreases as the floor value rises, since put options with higher strike prices must be purchased for the insurance, and they cost more. There is no investment in the riskless asset in the portfolio insurance portfolios.

Table I

			----- Portfolio Insurance Portfolios -----			--- Optimal Power-Log Portfolios ---		
Portfolio	Floor (%)	Expected Return (%)	Riskless Weight (%)	Stock Weight (%)	Put Weight (%)	Riskless Weight (%)	Stock Weight (%)	ATM Put Weight (%)
1	80	12.23	0.00	99.24	0.76	-30.58	124.53	6.04
2	85	11.83	0.00	98.56	1.44	-24.92	119.04	5.88
3	90	11.26	0.00	97.53	2.47	-16.93	111.31	5.62
4	95	10.56	0.00	96.12	3.88	-6.74	101.47	5.27

The last panel in Table I, "Optimal Power-Log Portfolios," shows the optimally hedged portfolios constructed by using Power-Log utility functions. The optimal Power-Log portfolio in each row of the table has been constructed to match the expected return of the portfolio insurance portfolio in the same row. The optimal hedge is created by using the riskless asset, the stock index and the ATM put option. The resulting optimally hedged portfolios for all the floor values in the table are leveraged with borrowing at the riskless rate. Investment in the put option is preferred to lending at the riskless rate for reducing downside exposure, since the put return is significantly positively skewed. Interestingly, as the floor value decreases from 95% to 80%, both the optimal leverage from borrowing at the riskless rate and the investment in the ATM put option increase. The increase in leverage pushes up expected portfolio return, while the increase in the ATM put option's weight controls downside exposure.

Table II shows the Value at Risk (VaR) and Value to Gain (VtG) at a 95% confidence level. Value to Gain is a measure for the potential value being added on the upside (Kale [12]). At a 95% confidence level if VtG is $1 million, then there is a 5% probability of a gain of $1 million or higher. The VaR and VtG are measures of location for the tails of the distribution and can be used for dollar, or percent losses and gains.

Table II

			Value at Risk (%)		Value to Gain (%)		VtG / VaR	
Portfolio	Floor (%)	Expected Return (%)	Portfolio Insurance	Power-Log	Portfolio Insurance	Power-Log	Portfolio Insurance	Power-Log
1	80	12.23	20.61	12.20	52.51	59.59	2.55	4.88
2	85	11.83	16.23	11.25	51.46	57.03	3.17	5.07
3	90	11.26	12.22	9.95	49.89	53.46	4.08	5.37
4	95	10.56	8.68	8.37	47.73	48.93	5.50	5.85

The VaR of the optimally hedged Power-Log portfolios shown in Table II is smaller than that of the corresponding portfolio insurance portfolios for every floor value shown in the table. The difference is particularly striking for the lower floor values. The VtG for the Power-Log portfolios is also consistently higher than that of the corresponding portfolio insurance portfolios for every floor value. Overall, at each level of expected return, the risk and asymmetry characteristics of optimally hedged portfolios are consistently superior to those of the corresponding portfolio insurance portfolios.

The last panel of Table II shows the ratio of VtG to VaR. The ratio shows the size of the potential gains relative to the size of the potential losses at the extremes. It is different from skewness, since the skewness measure does not focus on the tails of the distribution exclusively, as does the VtG to VaR ratio. The Power-Log portfolios have consistently higher VtG to VaR ratios than the corresponding portfolio insurance portfolios. The difference in the ratio is greater for the lower floor values, since the VaR numbers for the optimally hedged Power-Log portfolios are significantly smaller than those for the corresponding portfolio insurance portfolios.

Table III compares the more traditional asymmetry and risk characteristics of the return distributions of the portfolios. The skewness of the Power-Log portfolio returns is greater than that of the portfolio insurance portfolios for every floor value shown in the table. The negative semideviation is the semideviation for returns below zero, and is a measure of downside risk (Markowitz [19]). It is consistently lower for the optimally hedged Power-Log portfolios than for the corresponding portfolio insurance portfolios.

Table III

			Skewness		Neg. Semidev. (%)	
Portfolio	Floor (%)	Expected Return (%)	Portfolio Insurance	Power-Log	Portfolio Insurance	Power-Log
1	80	12.23	0.74	1.26	7.23	5.86
2	85	11.83	0.85	1.27	6.56	5.44
3	90	11.26	0.99	1.29	5.65	4.86
4	95	10.56	1.16	1.30	4.56	4.15

CONCLUSION

In this study we compare portfolios constructed by using a portfolio insurance strategy implemented with European put options with optimally hedged portfolios constructed by using an expected utility maximization with Power-Log utility functions. We use a riskless asset, a stock index and put options on the stock index to construct the portfolios. For a series of different floor values for the portfolio insurance portfolios, we construct optimally hedged portfolios that match the expected returns of the portfolio insurance portfolios. We find that the optimally hedged portfolios have return distributions with risk and asymmetry characteristics that are consistently superior to those for the return distributions of the corresponding portfolio insurance portfolios with the same expected return.

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