# A NEW METHOD OF SOLVING REVERSE CONVEX PROGRAM WITH APPLICATIONS IN LOGISTICS 

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#### Abstract

A new method for solving Reverse Convex Programs with applications in Supply Chain / Logistics is introduced. The method is cutting plane based and uses a variation of cuts introduced in [4]. Cutting planes traditionally have been used as a valuable tool in devising exact algorithms for solving large-scale combinatorial optimization problems. A cutting plane can be used to effectively reduce the computational efforts in search of a global solution. Each cut is generated in order to eliminate a large portion of the search domain. Thus, a deep cut is intuitively superior in that it will exclude a larger set of points from consideration. An upper bound and a lower bound for the optimal value is found and improved at each iteration. The algorithm terminates when all the generated partitions have been fathomed.


## INTRODUCTION

In many areas of logistics, there are two or more players who compete for the scarce resources such as facility space, transport vehicle, as well as goods and services. In certain problems, one player or agent has gained certain advantages (e.g., early market entry, or monopoly of important resources) that allow the player to act as a leader with a power to make the first decision. The rest of the players, or followers, observe the leader's decision and make their own decisions based on their own utility. In this specialized game, the leader cannot make its decision without any regards to the consequences of such decision may have on other player. That is, it may be harmful if the leader, by its decisions, forces the other players out of the market. One example of such game is a governmental agencies acting as a leader with the public welfare as its objective making tax decisions in order to curb firms' pollution output. These firms, as profit takers, are willing to absorb these taxes up to a limit but will leave the area if the leader's tax decisions are detrimental to their operations. This game can be modeled as a bilevel programming problem (BLPP) where the leader has an optimization problem of its own and imbedded in this optimization is the follower's optimization problem. We can denote the problem as,

$$
\begin{equation*}
\min \{F(x, y) \mid \min \{f(x, y) \mid g(x, y) \leq 0, x, y \geq 0\}\} \tag{1}
\end{equation*}
$$

It has been shown [1] that we can convert this problem to a single level problem with a reverse convex constraint:

$$
\begin{equation*}
\min \left\{c^{\circ} x \mid A x \leq b, g(x) \leq 0, x \geq 0\right\} \tag{P}
\end{equation*}
$$

where in the above setting $g: R^{n} \longrightarrow R^{1}$ is a concave function of $x$. We define $P=\{x \mid A x \leq b, x \geq 0\}$ to denote the polyhedral space where $A$ is a real $m \times n$ matrix and $c$ and $b$ are real vectors of order $n$ and $m$, respectively. Additionally, let $G=\{x \mid g(x) \leq 0\}$ and $F=P \cap G$ to denote the feasible domain for problem ( P ). We also assume that $F \neq \phi$ and that $P$ is bounded. Problem ( P ) is known to possess multiple local optimal solutions that are not globally optimal and hence the reason for its classification as a global optimization problem. It is well known that $(\mathrm{P})$ is mathematically intractable and in principle, a very difficult problem to solve. It has been shown to belong to class of NP-hard problems ([2], [3], [4]).

## PRELIMINARIES AND NOTATION

Consider problem $(\mathrm{P})$ and let $x^{0}=\operatorname{Argmin}\left\{c^{\circ} x \mid x \in P\right\}$ solve the associated linear program, herein referred to as (LP). Assume $x^{0}$ is a non-degenerate vertex with $n$ neighboring vertices of $x^{1}, x^{2}, \ldots, x^{n}$. Denote by $N\left(x^{0}\right)$, the set of neighboring vertices of $x^{0}$, let $x^{j} \in N\left(x^{0}\right)$ and $z^{j}=\left(x^{j}-x^{0}\right)$. That is $z^{j}$ is the direction of a ray from $x^{0}$ to $x^{j}$, its neighboring vertex. It is easy to extract items such as $z^{j}$ for every $x^{j} \in N\left(x^{0}\right)$ from the optimal simplex tableau of the linear program. That is, the tableau associated with the basic feasible solution $x^{0}$ contains these parameters. Let $x^{j}$ be an arbitrarily member of $N\left(x^{0}\right)$ and denote by $y^{j}$, the intersection of boundary of $g(x)$ or $\partial g$ and the ray $x^{0}+\alpha z^{j}$. The points $\left\{y^{j}, j=1, \ldots, n\right\}$ are found by solving the subproblem ( $\mathrm{SP}^{j}$ ) given below.

$$
\begin{equation*}
\min \left\{\alpha \mid 0 \leq \alpha \leq 1, g\left(x^{0}+\alpha\left(x^{j}-x^{0}\right)\right)=0\right\}, \quad\left(\mathrm{SP}^{j}\right) \tag{3}
\end{equation*}
$$

Additionaly, let $\bar{x}$ solve (CP) a related convex problem defined as:

$$
\begin{equation*}
\max \left\{c^{\circ} x \mid A x \leq b, g(x) \geq 0, x \geq 0\right\} \tag{CP}
\end{equation*}
$$

Clearly, $\bar{x}$ is feasible for ( P ) and thus it can be used to establish an initial upper bound. Furthermore, let $z=\left(\bar{x}-x^{0}\right)$ denote the ray passing through $\bar{x}$ and $x^{0}$ and for each vertex $x^{j} \in N\left(x^{0}\right)$ we let $z^{j}=\left(y^{j}-x^{0}\right)$ to denote a ray passing through $y^{j}$ and $x^{0}$. The $y^{j}$ is found by solving subproblem ( $\mathrm{SP}^{j}$ ) as described above. Since $x^{0}$ was assumed to be non-degenerate, such $z^{j}$ 's are linearly independent. Assume $z \neq z^{j}$ for $j=1, \ldots, n$ and construct the hyperplane $H_{j}=\left\{x \mid e^{\cdot} D_{j}^{-1}\left(x-x^{0}\right)=1\right\}$ and $H=\left\{x \mid e^{\cdot} D^{-1}\left(x-x^{0}\right)=1\right\}$ where

$$
\begin{gather*}
D_{j}=\left[z^{1}, z^{2}, \ldots, z^{j-1}, z, z^{j+1}, \ldots, z^{n}\right], \quad j=1, \ldots, n \text { and }  \tag{5}\\
D=\left[z^{1}, z^{2}, \ldots, z^{j-1}, z^{j}, z^{j+1}, \ldots, z^{n}\right] . \tag{6}
\end{gather*}
$$

The $H_{j}$ 's and $H$ are known as convexity cuts or Tuy cuts, passing through $\bar{x}$ and $y^{k}$, $k=1, \ldots, n ; k \neq j$ and $y^{k}, k=1, \ldots, n$, respectively. Each $H_{j}$ splits $P$ into two smaller polyhedral regions $\left\{A x \leq b, e^{\cdot} D_{j}^{-1}\left(x-x^{0}\right) \geq 1\right\}$ and $\left\{A x \leq b, e^{\cdot} D_{j}^{-1}\left(x-x^{0}\right)<1\right\}$. The point $\bar{x}$ may be thought of as a polar point of the convex region with the maximum distance from $x^{\circ}$ on the boundary of $g$ in the direction of $c$. The cuts $H_{j} ; j=1, \ldots, n$ and $H$ split $P$ into $(n+1)$ smaller polytopes, $P_{j}$, $j=1, \ldots, n$, where $P=P_{1} \cup P_{2} \cup \mathrm{~L} \cup P_{n} \cup P_{n+1}$ and $P_{i} \cap P_{j}=\phi$, for $i \neq j$ and $i, j \in\{1,2, \ldots, n+1\}$. The region, $P_{n+1}$, is contained in a cone $C$ in $R^{n}$ with $\bar{x}$ as its vertex and $\left(y_{0}^{j}-\bar{x}\right)$ 's as its rays. Aside from $\bar{x}$ and maybe some $y^{j}$ 's, $P_{n+1}$ does not contain any feasible points of (P) and thus will excluded from considerations. These polytopes are described as

$$
\begin{array}{lr}
P_{j}=\left\{x \mid x \in P, e^{\cdot} D_{j}^{-1}\left(x-x^{0}\right) \geq 1\right\} & \forall j=1, \ldots, n . \\
P_{n+1}=\left\{x \mid x \in P, e^{\cdot} D_{j}^{-1}\left(x-x^{\circ}\right) \leq 1,\right. & j=1, \mathrm{~L}, n\} . \tag{8}
\end{array}
$$

Also let $u^{j}, j=1, \ldots, n$ solve

$$
\begin{equation*}
\min \left\{c^{\cdot} x \mid A x \leq b, e^{\cdot} D_{j}^{-1}\left(x-x^{0}\right) \geq 1, x \geq 0\right\} \tag{9}
\end{equation*}
$$

for $j=1, \ldots, n$, respectively. If for any of the $u^{j}$ s, say, $u_{1}^{j}, g\left(u_{1}^{j}\right) \leq 0$, i.e., $u_{1}^{j}$ is feasible for (P), then this implies that $y^{j}=u_{1}^{j}$ and that it is a local solution to $(\mathrm{P})$.

If the process does not terminate or none of the $y_{k}^{j}$ is found feasible for $(\mathrm{P})$ then the hyperplanes $H_{j}^{k}$ 's must be updated and $H^{k}$ is moved deeper (in the direction of $c$ ) and the new cut is appended to P in the following manner. Compute the vertical distance $d_{k}^{j}$ from $u_{k}^{j}$ for $j=1, \ldots, n$ to the cut $H^{k}$. Find index 1, if some of the $y_{k}^{j}$ 's are feasible. If none were feasible, set $1=\infty$. It is easy to show that the minimum distance of the $u_{k}^{j}$ from the cut $H^{k}$ is given by $d_{k}^{j}$ where,

$$
\begin{equation*}
d_{k}^{j}=\frac{\left|1-e^{\cdot} D^{k^{-1}}\left(u_{k}^{j}-x^{0}\right)\right|}{\left\|e^{\cdot} D^{k^{-1}}\right\|} \tag{10}
\end{equation*}
$$

and $\|\cdot\|$ denotes the vector norm. Let denote the shortest vertical distance of all $u_{k}^{j}$ 's by $d_{k}$,

$$
\begin{equation*}
d_{k}=\min \left\{d_{k}^{j} \mid j=1, \ldots, n, \text { and } j \neq 1\right\} . \tag{11}
\end{equation*}
$$

Accordingly, $H^{k}$ is translated by amount $d_{k}$, in the direction of $c$, to produce a deeper cut $\hat{H}^{k}$.

$$
\begin{equation*}
\hat{H}^{k}=\left\{x \mid e^{\cdot} D^{k^{-1}} x=e^{\cdot} D^{k^{-1} u_{k}^{1}}\right\} \tag{12}
\end{equation*}
$$

The iteration counter is incremented by 10 and the process continues. It is well known that no feasible point of $F$ will be eliminated by introduction of the Tuy Cut $H^{k}$. On the other hand, $\hat{H}^{k}$ which is a translation of $H^{k}$ is deeper by the value of $d_{k}$ and by construction dominant with respect to $H^{k}$. By construction $P^{k+1} \subseteq P^{k} \subseteq \mathrm{~L} \subseteq P$ and the sequence of $\left\{x_{k}^{0}\right\}$ is such that $c^{\cdot} x_{k}^{0} \leq c^{\cdot} x_{k+1}^{0}$ for any iteration $k$. Furthermore, it is easy to show that the sequences of $\left\{U B_{k}\right\}$ and $\left\{L B_{k}\right\}$, are monotonically decreasing and increasing, respectively with a limiting value at $x^{\mathfrak{a}}$.

## Convergence

The proof of convergence is similar to that given in the Lemmas 1 and 2 of [4]. By the updating rules imposed on the hyperplanes $H^{k}, \hat{H}^{k}$, and $H_{j}^{k}$, the sequences of $\left\{c^{\cdot} y_{k}^{j}\right\}$ and $\left\{c^{\cdot} u_{k}^{j}\right\}$ for $k=1,2, \ldots$ are monotonically increasing and clearly bounded and thus convergent.

## CONCLUDING REMARKS

This paper offers a modification of the method presented in [4] to derive deeper cuts than that of the original concavity cuts. The algorithm also differs from that of [4] in that the branching process is executed at each iteration of the algorithm instead of taking place only once. This may seem more expensive than deriving an original concavity cut. But several studies and experiments with cutting planes in the context of cone decomposition have shown that it may be worth the added expenses if the derived cut is much deeper than the corresponding concavity cut. It is also worth noting that the number of cuts required overall may be much smaller than that with concavity cuts. This of course, remains to be investigated in computational experiments and follow up research.

## REFERENCES

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