# ASSET ALLOCATION WITH POWER-LOG UTILITY FUNCTIONS VS. MEAN-VARIANCE OPTIMIZATION

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## ABSTRACT

This study compares optimization with Power-Log utility functions with mean-variance optimization in a strategic asset allocation framework. It uses major asset classes to show that optimal Power-Log utility portfolios have lower downside risk and higher upside potential than mean-variance efficient portfolios. It also shows that intermediate term bonds should be given greater weight in conservative portfolios than mean-variance optimization suggests.

## INTRODUCTION

Strategic asset allocation is the process of selecting asset classes and their relative weights to construct optimal portfolios that are held over long time horizons. The mean-variance framework for portfolio selection developed by Markowitz [23] is a one-period model that is used widely for asset allocation, but there are other methods for portfolio selection. Multiperiod portfolio theory based on log and power utility functions has been discussed by Kelly [16] and others. Behavioral finance gives us a different perspective on investor actions based on prospect theory, proposed by Kahneman and Tversky [14] and Tversky and Kahneman [35], where they define utility separately over gains and losses. Kale [15] combines the methods of multiperiod portfolio theory with some of the tenets of prospect theory to develop Power-Log utility functions that balance growth maximization with downside protection, and shows the significant superiority of Power-Log utility optimization over mean-variance optimization when options are included in the portfolio. This study uses the same methodology for constructing portfolios with major asset classes, and compares them to mean-variance efficient portfolios.

## METHODOLOGY

Portfolio selection with a utility function uses the expected utility criterion developed by Von Neumann and Morgenstern [37] and Savage [34]. The description of the log, power and Power-Log utility functions that follows is based on Kale [15]. Each Power-Log utility function is a two-segment utility function, where the utility of gains is modeled with a log utility function and the utility of losses is modeled with a power utility function with power less than or equal to zero. It combines the maximum growth characteristics of the log utility function on the upside, with the scalable downside protection characteristics of the power function on the downside. It is defined as,

$$\begin{aligned} U &= \ln(1+r) \quad \text{for } r \geq 0 \\ &= \frac{1}{\gamma}(1+r)^{\gamma} \quad \text{for } r < 0 \end{aligned} \tag{1}$$

where,

- r portfolio return
- $\gamma$ power, is less than or equal to 0

Power-Log utility functions conform to the Kahneman and Tversky postulates of reference dependence, loss aversion, and diminishing sensitivity for gains. Investors can vary the level of downside protection they build into their portfolios by changing the downside power. Selecting a downside power of zero is equivalent to using a log utility function for losses, which will result in the construction of the maximum growth portfolio, since the utility function for gains is always a log utility function. Lower values of the downside power represent greater loss aversion since the penalty for losses increases, while the value associated with gains is left unchanged.

Another interesting characteristic of Power-Log utility functions is that they are continuously differentiable across the entire range of returns, which allows the development of fast optimization algorithms for portfolio selection. The algorithm used for this study is a nonlinear mathematical programming algorithm based on an accelerated conjugate direction method developed by Best and Ritter [1], and has a superlinear rate of convergence.

For a given Power-log utility function the optimal portfolio is selected by maximizing the expected utility of the portfolio. It requires the specification and use of the entire joint distribution of asset returns. As a result all the moments of the distribution of asset returns, including mean, variance, skewness, kurtosis and all the correlations between asset returns are implicitly taken into account.

The quarterly returns data used for constructing the portfolios is calculated from the monthly returns in Ibbotson Associates Yearbook [13]. All optimizations are constrained so that no short sales are permitted. The next section compares the portfolio compositions and the return characteristics of portfolios constructed by using the two methods.

## OPTIMAL PORTFOLIOS

Table I shows the portfolios constructed by using Power-Log utility functions. All the return and risk values shown in the tables have been annualized.

Table I. Optimal Power-Log Portfolio Asset Weights

| No. | Downside Power | Expected Return (%) | T-Bill Weight (%) | IT Treas. Weight (%) | LT Treas. Weight (%) | S&P500 Weight (%) |
|---|---|---|---|---|---|---|
| 1 | 0 | 13.99 | 0.00 | 0.00 | 0.00 | 100.00 |
| 2 | -2 | 12.38 | 0.00 | 18.08 | 0.00 | 81.92 |
| 3 | -4 | 10.22 | 0.00 | 42.69 | 0.00 | 57.31 |
| 4 | -10 | 8.06 | 0.00 | 67.74 | 0.00 | 32.26 |
| 5 | -20 | 7.04 | 0.00 | 79.69 | 0.00 | 20.31 |
| 6 | -30 | 6.37 | 14.78 | 70.04 | 0.00 | 15.18 |
| 7 | -40 | 5.93 | 27.24 | 60.54 | 0.00 | 12.22 |
| 8 | -50 | 5.65 | 35.21 | 54.48 | 0.00 | 10.31 |

The optimal Power-Log portfolio in Row 1 of Table I has been constructed with a downside power of 0 and is the maximum growth portfolio, which consists of a 100% investment in stocks. In retirement fund applications an all stock portfolio is typically recommended to very young investors, and as the age of investors increases more conservative portfolios with larger holdings in fixed income assets are recommended. Table I shows that as the downside power is lowered, more downside protection is built into the portfolios and the resulting portfolios are progressively more conservative, contain greater proportions of fixed income assets and consequently have lower expected returns. Interestingly, long term treasuries do not appear in any of the optimal portfolios. Instead, intermediate term treasuries have the largest weightings in all the optimal portfolios for downside powers of -10 and lower. These optimal portfolios suggest that the appropriate asset mix for most investors consists of stocks and intermediate term treasuries, with significant additions of T-Bills for the most conservative investors only.

Table II shows the mean-variance efficient portfolios that have been constructed to match the expected return of the optimal Power-Log portfolios in Table I. The expected portfolio return has been selected as the point of reference for comparing the different techniques of portfolio construction, since it is appropriate as a measure of reward in all cases, while other reference measures such as standard deviation are inappropriate when downside risk is at issue. The portfolio with the highest expected return is an all stock portfolio just like the corresponding optimal Power-Log portfolio; both reflect the substantially higher returns to stocks relative to treasuries over long histories. For the mean-variance efficient portfolios shown in rows one through four, the asset weights are very similar to those for the corresponding optimal Power-Log portfolios. However, the more conservative mean-variance efficient portfolios in rows five through eight have a substantially higher weighting for T-Bills, and a substantially lower weighting for intermediate term treasuries than the corresponding optimal Power-Log portfolios. The greater longer term benefit of owning intermediate term treasuries instead of T-Bills is not picked up in mean-variance optimization as well as it is with Power-Log utility optimization.

Table II. Mean-Variance Efficient Portfolio Asset Weights

| No. | Expected Return (%) | T-Bill Weight (%) | IT Treas. Weight (%) | LT Treas. Weight (%) | S&P500 Weight (%) |
|---|---|---|---|---|---|
| 1 | 13.99 | 0.00 | 0.00 | 0.00 | 100.00 |
| 2 | 12.38 | 0.00 | 18.23 | 0.00 | 81.77 |
| 3 | 10.22 | 0.00 | 42.90 | 0.00 | 57.10 |
| 4 | 8.06 | 0.00 | 68.00 | 0.00 | 32.00 |
| 5 | 7.04 | 8.96 | 69.35 | 0.00 | 21.69 |
| 6 | 6.37 | 27.60 | 55.09 | 0.00 | 17.31 |
| 7 | 5.93 | 40.09 | 45.54 | 0.00 | 14.37 |
| 8 | 5.65 | 48.09 | 39.42 | 0.00 | 12.49 |

Tables III and IV show some summary statistics for the portfolios constructed by using the two different methods. As expected, the standard deviation for the mean variance efficient portfolios is slightly lower than that for the optimal power-Log portfolios, since mean-variance optimization explicitly minimizes the variance to construct an optimal portfolio for a given expected return. However, the Minimum Return and Value at Risk (VaR) shown in the tables are better measures of the downside risk associated with the portfolios. These measures have been calculated for all the optimal portfolios by bootstrapping the quarterly joint return distribution of asset returns. The Minimum Return for the optimal Power-Log portfolios is higher than that for all the corresponding mean-variance efficient portfolios, except for the portfolios in Row 1 where it is the same. The biggest difference in Minimum Return is for the portfolios in Row 8, the most conservative portfolios, where it is 2.1% higher for the optimal Power-log portfolios.

The same pattern of differences holds for VaR, which has been calculated at the 95% confidence level. These results suggest that the more conservative investors would be better off using the optimal Power-Log portfolios than the mean-variance efficient ones since they provide better protection from losses, which is an important goal for conservative investors. Alternatively, investors concerned about downside risk can invest in a portfolio that has a higher expected return than a mean-variance efficient portfolio with the same downside risk. Since small increases in annual expected return compound to significant increases over long horizons, this is a clear advantage for the optimal Power-Log portfolios.

Table III. Optimal Power-Log Portfolio Summary Statistics

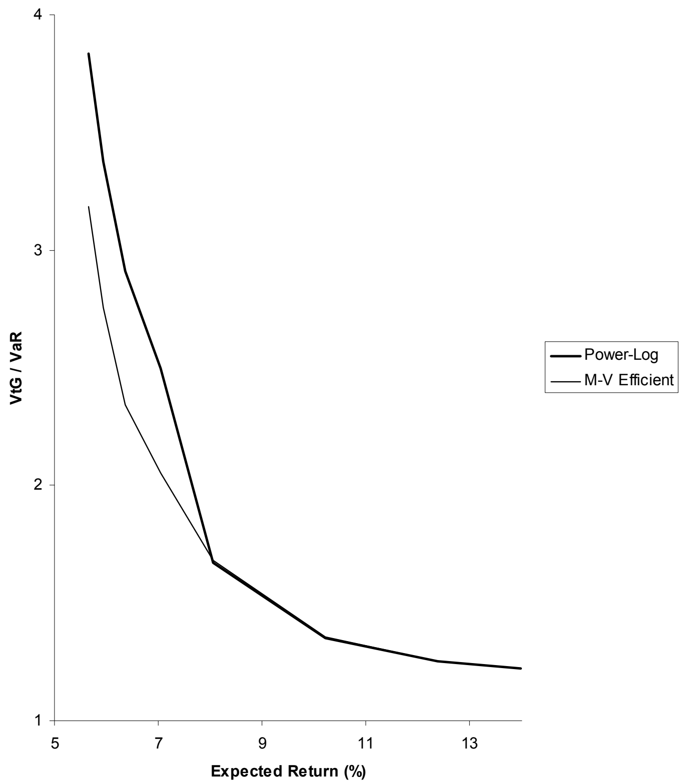
| No. | Downside Power | Expected Return (%) | Standard Deviation (%) | Minimum Return (%) | VaR (%) | VtG (%) | VtG/VaR |
|---|---|---|---|---|---|---|---|
| 1 | 0 | 13.99 | 23.21 | -84.92 | 69.34 | 88.53 | 1.22 |
| 2 | -2 | 12.38 | 19.14 | -76.65 | 55.50 | 72.63 | 1.25 |
| 3 | -4 | 10.22 | 13.72 | -60.45 | 37.58 | 53.06 | 1.35 |
| 4 | -10 | 8.06 | 8.58 | -38.34 | 20.36 | 35.67 | 1.67 |
| 5 | -20 | 7.04 | 6.54 | -26.69 | 11.21 | 29.69 | 2.49 |
| 6 | -30 | 6.37 | 5.31 | -20.76 | 8.31 | 25.60 | 2.91 |
| 7 | -40 | 5.93 | 4.49 | -17.03 | 6.45 | 23.00 | 3.37 |
| 8 | -50 | 5.65 | 3.97 | -14.58 | 5.16 | 20.88 | 3.83 |

Table IV. Mean-Variance Efficient Portfolio Summary Statistics

| No. | Expected Return (%) | Standard Deviation (%) | Minimum Return (%) | VaR (%) | VtG (%) | VtG/VaR |
|---|---|---|---|---|---|---|
| 1 | 13.99 | 23.21 | -84.92 | 69.34 | 88.53 | 1.22 |
| 2 | 12.38 | 19.11 | -76.57 | 55.40 | 72.49 | 1.25 |
| 3 | 10.22 | 13.67 | -60.28 | 37.40 | 52.88 | 1.35 |
| 4 | 8.06 | 8.53 | -38.10 | 20.22 | 35.64 | 1.68 |
| 5 | 7.04 | 6.47 | -27.86 | 12.95 | 27.95 | 2.06 |
| 6 | 6.37 | 5.21 | -22.68 | 9.47 | 23.20 | 2.34 |
| 7 | 5.93 | 4.38 | -19.06 | 6.98 | 20.10 | 2.75 |
| 8 | 5.65 | 3.86 | -16.68 | 5.47 | 18.17 | 3.18 |

Tables III and IV also show the Value to Gain (VtG), which is a measure of upside potential as described in Kale [15]. The Value to Gain is complementary to Value at Risk. At a 95% confidence level, if VtG is 20.88%, then there is a 5% probability of a gain of 20.88% or higher. The VtG for the optimal Power-Log portfolios is higher than that for all the corresponding mean-variance efficient portfolios, except for the portfolios in Row 1 where it is the same. Thus not only do the optimal Power-Log portfolios provide better downside protection, they also provide better upside potential than the corresponding mean-variance efficient portfolios. Figure 1 compares the VtG/VaR ratios for the two sets of optimal portfolios at a 95% confidence level, and shows the superiority of the more conservative optimal Power-Log portfolios.

Figure 1. VtG / VaR Ratio for Optimal Portfolios




## CONCLUSION

Strategic asset allocation is the process of selecting asset classes and their relative weights to construct optimal portfolios that are held over long time periods. This study examines the effectiveness of using Power-Log utility functions, in constructing portfolios from major asset classes consisting of large-company stocks, long term treasuries, intermediate term treasuries and T-bills. It compares the optimal Power-Log portfolios with the corresponding mean-variance efficient portfolios that have matched expected returns. While the composition and characteristics of the maximum growth portfolio produced by Power-Log utility optimization is the same as that of the corresponding mean-variance efficient portfolio, the more conservative optimal Power-Log portfolios consistently have lower downside risk and greater upside potential than the corresponding mean-variance efficient portfolios with the same expected return. This study also shows that for even the most conservative portfolios, the majority of the fixed income component of an asset allocation should consist of intermediate terms bonds and not T-Bills, when the goal is long term growth coupled with downside protection.

## ENDNOTES

1. For a detailed literature survey and references, please contact the authors.