A NEW DYNAMIC CHARACTERIZATION OF THE COMPLETELY COMPETITIVE EQUILIBRIA IN N-FIRM OLIGOPOLIES

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ABSTRACT

An approximate gradient adjustment process is introduced as realistic behavior of the firms in *n*-firm oligopolies, where the firms know their own cost functions and are able to observe the market price at each time period. The steady state of the resulting dynamic system is the completely competitive equilibrium, where the marginal cost of each firm equals the market price. The asymptotic stability of the steady state is proved under realistic conditions, which are much weaker in the continuous case than under discrete time scales.

INTRODUCTION

Since the pioneering work of Cournot (1838) a great number of scientists contributed to the interesting field of oligopoly theory. The existence and uniqueness of the Cournot-Nash equilibrium was the central issue in the early studies, and then different static and dynamic variants and extensions became more important to study. A comprehensive summary of early works can be found in Okuguchi(1976) and their multi-product generalizations with several applications are discussed in Okuguchi and Szidarovszky(1999). An excellent summary of nonlinear phenomena in economics is presented in Puu(2000). In most works complete information was assumed to all firms, that is, it was assumed that all firms knew the inverse demand function and all cost functions, therefore they were able to compute their best responses. The effect of the inexact knowledge of the inverse demand function on the asymptotic properties of dynamic oligopolies was examined only by a few authors. Leonard and Nishimura(1999) have assumed discrete time scales and that the shape of the price function is know by all firms, but its scale is uncertain. They studied how the asymptotic behavior of the resulting dynamic systems changed in comparison to the full information case. The continuous counterpart of the Leonard-Nishimura model was examined in Chiarella and Szidarovszky(2001, 2004), where a time delay was also assumed in the reaction functions of the firms. In these models it was assumed that each firm had its own estimate of the inverse demand function, and this estimate was used to obtain the believed best responses in the dynamic process. The output strategies were shown to converge to the Cournot-Nash equilibrium or to a “believed” equilibrium of the static oligopoly game.

In this paper we do not even assume the existence of believed inverse demand function, we assume only that the firms are able to observe the received price at each time period, but each firm knows its own cost function. Therefore they cannot obtain even an estimate of their best responses, however they are able to estimate their marginal profits and therefore an approximate gradient process can develop. We will show that under realistic conditions this process converges to the completely competitive equilibrium. From the literature (see for example, Okuguchi, 1976) we also know that it can be obtained as the limit of oligopoly equilibria if the number of firms tends to infinity under certain conditions. This result however was criticized by many authors by the occurrence of negative prices. Our model however presents a realistic dynamic process with the same limiting property.

THE MATHEMATICAL MODEL

Let n denote the number of firms producing the same product or offering identical service. If x_k is the output of firm k, $C_k(x_k)$ is its cost function, and $f(s)$, with $s = \sum_{k=1}^{n} x_k$, is the inverse demand function, then the profit of firm k can be written as

$$\varphi_k(x_1,...,x_n) = x_k f(\sum_{l=1}^{n} x_l) - C_k(x_k).$$

It is assumed that each firm knows its own cost function, but the inverse demand function-as a function –is unknown, only its value can be observed at each time period. It is also assumed, that the firms are unable to observe the outputs of the competitors, therefore the repeated price observations cannot lead to a reliable estimate of the price function.

Firm k therefore estimates its marginal profit in the following way:

$$\frac{\partial \varphi_k}{\partial x_k} \approx \lim_{h \to 0} \frac{[(x_k + h) f(\sum_{l=1}^{n} x_l) - C_k(x_k + h)] - [x_k f(\sum_{l=1}^{n} x_l) - C_k(x_k)]}{h}$$

$$= f(\sum_{l=1}^{n} x_l) - C_k'(x_k), \tag{2}$$

since the function form of the price function is unknown, the firms cannot assess the value of $f(\sum_{l=1}^{n} x_l + h)$, only the observed value $f(\sum_{l=1}^{n} x_l)$ is known by the firms. If $K_k > 0$ is the speed of adjustment of firm k, then this first adjusts its output as

$$x_k(t+1) = x_k(t) + K_k(f(\sum_{l=1}^{n} x_l(t)) - C_k'(x_k(t))) \tag{3}$$

under discrete time scales and as

$$x_k'(t) = K_k(f(\sum_{l=1}^{n} x_l(t)) - C_k'(x_k(t))) \tag{4}$$

under continuous time scales. Note that systems (3) and (4) are approximate gradient adjustment processes.

An output vector $\bar{\mathbf{x}} = (\overline{x_1}, \ldots, \overline{x_n})$ is the steady state of system (3) (or (4) if and only if for all k,

$$f(\sum_{l=1}^{n} \overline{x_l}) = C_k'(\overline{x_k}), \tag{5}$$

that is, when $\bar{\mathbf{x}}$ is a completely competitive equilibrium.

ASYMPTOTIC BEHAVIOR OF THE STEADY STATE

The asymptotic properties of systems (3) and (4) are examined by linearization.

Consider first system (3). The Jacobian of the system has the special form

$$\mathbf{J}^D = \begin{pmatrix} 1+K_1(f^{'}-C_1^{''}) & K_1 f^{'} & \dots & K_1 f^{'} \\ K_2 f^{'} & 1+K_2(f^{'}-C_2^{''}) & \dots & K_2 f^{'} \\ \vdots & \vdots & \ddots & \vdots \\ K_n f^{'} & K_n f^{'} & \dots & 1+K_n(f^{'}-C_n^{''}) \end{pmatrix} \tag{6}$$

where all derivatives are computed at the steady state. Notice that

$$\mathbf{J}^D = \mathbf{D} + \mathbf{a}\cdot\mathbf{1}^T \tag{7}$$

where

$$\mathbf{D} = diag(1-K_1C_1^{''},\dots,1-K_nC_n^{''}),\ \mathbf{a} = (K_1 f^{'},\dots,K_n f^{'})^T$$

and

$$\mathbf{1}^T = (1,\dots,1)\,.$$

Therefore the characteristic polynomial of $\mathbf{J}^D$ has the special form

$$\begin{aligned}\varphi^D(\lambda) &= \det(\mathbf{D}+\mathbf{a}\cdot\mathbf{1}^T - \lambda\mathbf{I}) = \det(\mathbf{D}-\lambda\mathbf{I})\cdot\det(\mathbf{I}+(\mathbf{D}-\lambda\mathbf{I})^{-1}\mathbf{a}\cdot\mathbf{1}^T) \\ &= \prod_{k=1}^{n}(1-K_kC_k^{''}-\lambda)\cdot\left[1+\sum_{k=1}^{n}\frac{K_k f^{'}}{1-K_kC_k^{''}-\lambda}\right],\end{aligned} \tag{8}$$

when we used the well-known fact, that with any vectors $\mathbf{u},\mathbf{v}\in R^n$, $\det(\mathbf{I}+\mathbf{u}\cdot\mathbf{v}^T) = 1+\mathbf{v}^T\cdot\mathbf{u}$ (see for example, Okuguchi and Szidarovszky, 1999).

Let $\alpha_1 > \alpha_2 > \dots > \alpha_s$ denote the different $K_kC_k^{''}$ values and let

$$I_i = \left\{k \mid K_kC_k^{''} = \alpha_i\right\}.$$

By introducing the notation $m_i = |I_i|$ and $\beta_i = \sum_{k\in I_i} K_k$, we have

$$\varphi^D(\lambda) = \prod_{i=1}^{s}(1-\alpha_i-\lambda)^{m_i}\cdot\left[1+\sum_{i=1}^{s}\frac{\beta_i f^{'}}{1-\alpha_i-\lambda}\right]. \tag{9}$$

The eigenvalues of $\mathbf{J}^D$ are $1-\alpha_i$ for $m_i \geq 2$, and the roots of equation

$$\sum_{i=1}^{s}\frac{\beta_i f^{'}}{1-\alpha_i-\lambda} = -1\,. \tag{10}$$

If $g(\lambda)$ denote the left hand side of this equation, then clearly

$$\lim_{\lambda\to\pm\infty} g(\lambda) = 0,\ \lim_{\lambda\to 1-\alpha_i\pm 0} g(\lambda) = \pm\infty$$

and

$$g^{'}(\lambda) = \sum_{i=1}^{s}\frac{\beta_i f^{'}}{(1-\alpha_i-\lambda)^2} < 0\,.$$

Here we use the condition $f^{'} < 0$, which is the case in realistic economics, since f has to be strictly decreasing in the total output of the industry. The graph of $g(\lambda)$ is shown in the figure.

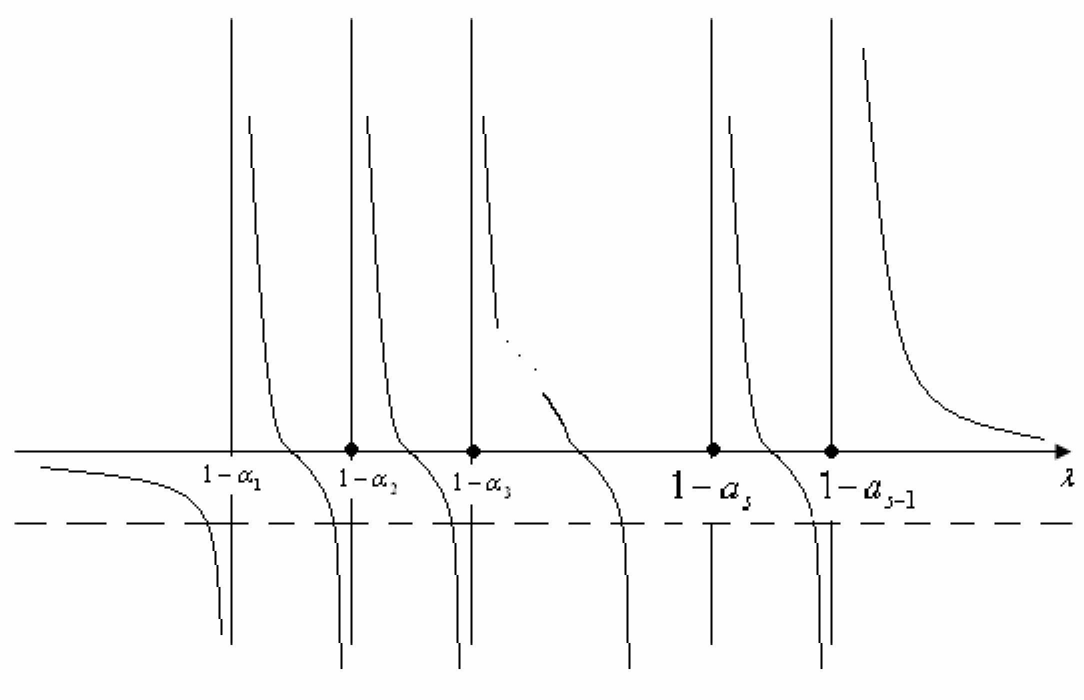

Figure. The graph of function $g(\lambda)$

Equation (10) is equivalent to a polynomial equation of degree s, and there is one root before $1-\alpha_1$ and one between each pair of poles $1-\alpha_i$ and $1-\alpha_{i+1}$ $(1 \le i \le s-1)$. Therefore all roots are real. The steady state is locally asymptotically stable if all eigenvalues of $\mathbf{J}^D$ are inside the unit circle. Therefore we have the following result.

Theorem 1 Assume that for all k,

$$0 < K_k C_k''(\overline{x_k}) < 2, \tag{11}$$

and

$$\sum_{k=1}^{n} \frac{K_k}{2 - K_k C_k''(\overline{x_k})} \cdot f'(\sum_{l=1}^{n} \overline{x_l}) > -1. \tag{12}$$

Then the steady state is locally asymptotically stable.

Proof Relation (11) implies that all eigenvalues of the form $1-\alpha_i$ are between -1 and +1 as well as all pole of function $g(\lambda)$ are also there. Therefore all roots of $g(\lambda)$ which are between the poles are also between -1 and +1, and the smallest root is above -1 if $g(-1) > -1$. This is equivalent to relation (12).

Remark 1 Condition (11) can be rewritten as

$$C_k''(\overline{x_k}) > 0 \text{ and } K_k < \frac{2}{C_k''(\overline{x_k})}, \tag{13}$$

so in the neighborhood of $\overline{x_k}$, the cost function has to be locally strictly convex, and the speed of adjustment of firm k has to be sufficiently small. Relation (12) has the equivalent form

$$\sum_{k=1}^{n} \frac{K_k}{2 - K_k C_k^{''}(\overline{x_k})} < -\frac{1}{f^{'}(\sum_{k=1}^{n} \overline{x_k})}, \tag{14}$$

which holds if and only if the spends of adjustment of all firms are sufficiently small. In the symmetric case, $K_k \equiv K$ and $C_k^{''}(\overline{x_k}) \equiv C^{''}$, and (14) reduces to the following:

$$K < \frac{2}{C^{''} - nf^{'}}. \tag{15}$$

That is, the common speed of adjustment of the firms has to be sufficiently small.

Turning our attention to the continuous case (4), notice first that the Jacobian is now the following:

$$\mathbf{J}^C = \begin{pmatrix} K_1(f^{'} - C_1^{''}) & K_1 f^{'} & \dots & K_1 f^{'} \\ K_2 f^{'} & K_2(f^{'} - C_2^{''}) & \dots & K_2 f^{'} \\ \vdots & \vdots & \ddots & \vdots \\ K_n f^{'} & K_n f^{'} & \dots & K_n(f^{'} - C_n^{''}) \end{pmatrix} \tag{16}$$

which has a similar form as in the discrete case:

$$\mathbf{J}^C = (\mathbf{D} - \mathbf{I}) + \mathbf{a} \cdot \mathbf{1}^T, \tag{17}$$

where $\mathbf{D}$, $\mathbf{a}$ and $\mathbf{1}^T$ are the same as before. The characteristic polynomial of $\mathbf{J}^C$ is now

$$\varphi^C(\lambda) = \prod_{k=1}^{n}(-K_k C_k^{''} - \lambda) \cdot [1 + \sum_{k=1}^{n} \frac{K_k f^{'}}{-K_k C_k^{''} - \lambda}], \tag{18}$$

and in the continuous case we have the following results.

<u>Theorem 2</u> Assume that for all k, $C_k^{''}(\bar{x}_k) > 0$. Then the steady state is always locally asymptotically stable.

<u>Proof</u> We can follow the proof of Theorem 1 with the only difference that in this case all numbers $-K_k C_k^{''}$ are negative, therefore all possible roots of the first factor of (18) and also all poles of the second factor are negative.

<u>Remark</u> The condition of the theorem requires that the cost functions are locally strictly convex in a neighborhood of the steady state.

CONCLUSIONS

In this paper *n*-firm oligopolies were examined under the assumption that the firms know their own cost functions and they are able to observe the market price at each time period. An approximating gradient adjustment process was introduced, the steady state of which is the completely competitive equilibrium. Under reasonable assumptions this steady state is locally asymptotically stable. In the continuous case the price function has to be strictly decreasing and the cost function strictly convex in the neighborhood of the steady state, and in the discrete cases in addition, the speeds of adjustment of the firms have to be sufficiently small.

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