

RANGE OF OPTIMALITY SENSITIVITY ANALYSIS IN PORTFOLIO SELECTION WITH SIMULTANEOUS MARKET VALUE CHANGES IN EXPECTED RETURNS

*Peter M. Ellis, Department of Management, Utah State University, Logan, UT, 84322-3500
435-797-2372, peter.ellis@usu.edu*

ABSTRACT

Linear programming is well-known in portfolio selection problems. Sensitivity analysis of objective function coefficients is used to establish when expected returns on a security change so much that a new portfolio must emerge. There has not been known heretofore a method that permits this sensitivity analysis when several expected return coefficients vary simultaneously. That is a serious restriction because return coefficients are most likely to fluctuate synchronously with changing market conditions. A new procedure is presented here that carries out the range of optimality sensitivity analysis for the optimal portfolio when expected returns are subject to synchronized fluctuations.

INTRODUCTION

There is a lengthy history of the application of linear programming and its extensions to the portfolio selection problem in finance. Objective functions might be either maximization or minimization. The goal might be to maximize expected return on the portfolio. It might be to minimize some measure of portfolio risk. Variations include maximin and minimax satisfying. Goal programming has been used where multiple objectives are desired. Markowitz (1952) [5] very early formulated the problem as the minimization of a quadratic risk function with side restrictions. We now describe it as a quadratic programming problem and solve it with special adaptations of linear programming.

Linear programming has been a commonly used procedure in portfolio selection research and practice for many years. Speranza (1993) [8] presented a variety of linear programming formulations related to portfolio selection. Gennotte and Jung (1994) [1] introduced the additional consideration of including transaction costs. Kellerer et al (2000) [2] extended that even farther by also including minimum transaction lots. Li and Deng (2000) [4] also developed a linear programming portfolio model that included transaction costs. Konno and Yamazaki (1991) [3] [replaced the Markowitz quadratic risk approach with a model of absolute deviation of returns from the expected value, then used linear programming to obtain an optimal solution. This was followed by Chiodi et al (2003), who also used absolute deviation to include portfolio risk. Ogryczak (2000) [6] used goal programming for the portfolio selection problem with multiple criteria. Young (1998) [9] sought a minimax portfolio selection solution with linear programming.

These linear programming approaches to the portfolio selection problem have examined problem formulation while including such ideas as risk, transaction costs and multiple investment goals. There has never been great attention given to post-optimality analysis, especially objective function sensitivity analysis. The probable reason for that is that expected returns found in the objective function are quite likely to vary in some simultaneous fashion in accordance with market conditions.

In linear programming the optimal solution is open to further investigation through range of optimality sensitivity analysis. In brief, it is known that if an objective function coefficient is changed too much either positively or negatively, the optimal solution may change to a different extreme point. In terms of

the investment problem, this means that a different portfolio mix might be called for. The analysis has always had the serious limitation that just one objective function coefficient change at a time can be considered. It has always been required that all other parameters of the problem be fixed and set while just the single objective function coefficient is permitted to vary. Of course, this is quite unrealistic in the portfolio selection problem with an objective of maximization of expected returns because the various individual returns on the candidate securities are very likely to fluctuate quite synchronously and in a predictable fashion. Changing market conditions cause security prices to change in response. The degree of commonality of response is heightened if the securities under consideration are closely tied to general market conditions.

SENSITIVITY ANALYSIS OF EXPECTED RETURNS IN THE OPTIMAL PORTOLIO

We consider a typical linear programming portfolio optimization problem where the objective is to maximize expected return on the overall investment. The constraints serve to force diversification preferences. For the purpose of the work developed here, the model is quite basic, not including such advanced considerations as short selling or transaction costs.

The linear programming formulation is:

$$\text{Minimize } Z = \sum_{i=1}^n C_i x_i \quad (\text{Ia})$$

Subject to:

$$\sum_{i=1}^n w_i x_i = 1 \quad (\text{Ib})$$

$$\sum_{i=1}^n a_{ij} x_i (\leq)(=)(\geq) R_j \quad j = 1, 2, \dots, M \quad (\text{Ic})$$

$$x_i \geq 0 \quad (\text{Id})$$

- n = number of securities being considered for the portfolio
- j = constraint number for any of the diversification requirements
- C_i = expected return on security i
- w_i = dollar percentage of the portfolio that is invested in security i
- a_{ij} = risk diversification coefficient for security i in risk requirement j
- R_j = limit on risk specification j
- x_i = percentage of the dollar amount of the portfolio to be held in security i
- M = number of constraint conditions upon the portfolio

The objective function (Ia) maximizes total percentage return of the portfolio. Constraint (Ib) requires that the dollar amount percentages of each security in the portfolio sum to 100%. The several constraints of the form (Ic) require that the risk and diversification requirements be satisfied. Last, the percentage allocations are required in (Id) to all be nonnegative.

The optimal solution identifies the dollar percentage allocation of each security to be included in the portfolio. After the solution is obtained, attention turns to the question of how the portfolio might change in response to different expected returns on any of the candidate securities. That inquiry has always been addressed by the postoptimality analysis known as the range of optimality. Briefly said, this sensitivity

analysis shows an interval for each expected return coefficient C_i wherein the optimal percentage portfolio allocations stay the same. The analysis requires that all other parameters of the problem remain fixed, including all other C_i values. Of course, the total expected return would change in consequence of the new C_i value for security i . The criticism of this has always been that the expected returns C_i are very unlikely to singly and independently vary over time. It would typically be found that expected returns are all subject to the same market conditions and influences, and would thereby vary in relative synchrony.

A method for carrying out the range of optimality analysis when the expected returns vary simultaneously will be shown here. It is limited to the important special case where the changes in the individual returns vary in constant ratio. One of the securities is chosen as the basis of comparison for fluctuations in expected return of the other securities. A beta coefficient b_i is calculated for every other expected return in comparison the expected return of the basis security. The b_i value = 1 for the basis security. Let D be the total expected change in expected return on the portfolio that arises from a 1% change in the expected return from the basis security. We have

$$D = \sum_{i=1}^n b_i x_i$$

A new constraint is formed by writing this as

$$D - \sum_{i=1}^n b_i x_i = 0 \tag{1e}$$

D is implicitly included in the objective function because it has a zero coefficient there. Also, D is not bound in any way in the constraints so its presence in the problem does not alter the outcome. However, it is important to include for the purpose of carrying out the range of optimality analysis. Because D is a variable of the problem, its range of optimality appears in the conventional linear programming sensitivity analysis report. An upper bound and a lower bound on the objective function coefficient of D will be reported in any linear programming software package. The bounds represent the maximum amount by which the expected return of the chosen basis variable can be permitted to increase or decrease without forcing the abandonment of the current optimal allocation and the consequential emergence of a new optimal allocation. Since the several expected return coefficients are linked through the associated beta coefficients, the result is that the sensitivity report for D shows the largest magnitude that each of the expected return coefficients can increase or decrease and still leave the current optimal portfolio unchanged.

AN EXAMPLE

Five shares are being considered for the portfolio: S&P500 Index, S&P Midcap Index, S&P Smallcap Index, S&P Value Index and S&P500 Growth Index. Monthly return data are from the 60 months from January 2004 to December 2008. The basis security is selected to be the S&P 500 Index. The other four securities are made the dependent variable as a linear regression is run for each with the S&P 500 Index serving as the independent variable in each case. The beta coefficient is 1.00 for the S&P 500 Index, 1.2146 for the Midcap Index, 1.1955 for the Smallcap Index, 1.0165 for the Value Index and .9850 for the Growth Index. It does seem anomalous that the growth index was less than one over this time period. The respective R^2 values were 1.00, .9013, .8172, .9455 and .9468. It thereby seems generally true that expected monthly returns in all the securities vary synchronously and in direct ratio to changes in the monthly return on the S&P 500. The last returns available were for December 2008. They were .01, .04, .05, .01 and .01, respectively. These returns will be used as the C_i values.

We seek to maximize the total dollar value of expected return on a portfolio that consists of these five securities. No security can have a dollar value in excess of 30% of the portfolio. The sum of the S&P 500 and the S&P Value Index portions must be at least as large as the sum of the S&P Smallcap Index and the S&P Growth Index portion. The sum of the S&P 500 Index and the S&P Midcap Index portion must be at least double the S&P Growth Index portion. The weighted average beta of the portfolio must be between 1.10 and 1.15. The linear programming formulation of the problem is shown below. The LINDO linear programming software of Schrage (1991) was used.

```

MAX    0.01 SP + 0.04 MID + 0.05 SMALL + 0.01 VALUE + 0.01 GROW
SUBJECT TO
WEIGHT) SP + MID + SMALL + VALUE + GROW = 1
MIX1) SP <= 0.3
MIX2) MID <= 0.3
MIX3) SMALL <= 0.3
MIX4) VALUE <= 0.3
MIX5) GROW <= 0.3
DIVERSE1) SP - SMALL + VALUE - GROW >= 0
DIVERSE2) SP + MID - 2 GROW >= 0
BETA1) SP + 1.2146 MID + 1.1955 SMALL + 1.0165 VALUE + 0.985 GROW
      <= 1.15
BETA2) SP + 1.2146 MID + 1.1955 SMALL + 1.0165 VALUE + 0.985 GROW
      >= 1.1
SENS) - SP - 1.2146 MID - 1.1955 SMALL - 1.0165 VALUE - 0.985 GROW + D
      = 0
END

```

The variables are, respectively, SP, MID, SMALL, VALUE and GROW. The several constraints are named in accordance with their purpose. The constraint SENS) has the form of (Ie). It establishes D as the total change in the expected return on the portfolio when the return on SP changes by $\pm 1\%$.

The optimal solution is:

LP OPTIMUM FOUND AT STEP 7

OBJECTIVE FUNCTION VALUE

1) .31000000E-01

VARIABLE	VALUE	REDUCED COST
SP	.050000	.000000
MID	.300000	.000000
SMALL	.300000	.000000
VALUE	.300000	.000000
GROW	.050000	.000000
D	1.127230	.000000

Also, the range of optimality sensitivity analysis is:

VARIABLE	OBJ COEFFICIENT RANGES		
	CURRENT COEF	ALLOWABLE INCREASE	ALLOWABLE DECREASE
SP	.010000	.000000	INFINITY
MID	.040000	INFINITY	.030000
SMALL	.050000	INFINITY	.040000
VALUE	.010000	INFINITY	.000000

GROW	.010000	.040000	.000000
D	.000000	.000000	.000000

We denote this original problem formulation and solution as stage 1. The optimal portfolio weights are, respectively, .05, .3, .3, .3 and .05. The expected overall return on the portfolio is 3.1%. D has an objective function coefficient of zero. The sensitivity report shows that the maximum increase and the maximum decrease for D are both equal to zero. Hence, any small increase or any small decrease in the zero coefficient will bring about a different optimal portfolio. We first consider a small increase.

Let the change in the expected return of the basic variable SP to be d . For each other security i the beta coefficient is b_i . The revised expected return coefficient for security i is then

$$C_{i \text{ new}} = C_i + d * b_i \tag{II}$$

Set the increase in the expected return on SP to be .001. Apply the beta coefficients to (II) and calculate new expected returns on each of the securities. For SP the new coefficient is $.01 + 1 * .001$. For MID the new expected return is $.04 + (.001) * (1.2146)$. The others are done in the same fashion. The expected returns are, respectively, .011, .0412, .0512, .011 and .0109. A linear programming model is created using the new expected return coefficients. Call this stage 2. The optimal portfolio for stage 2 has weights .1, .3, .3, .3 and 0, respectively. The expected return on the portfolio has risen to 3.231%.

At stage 1 the upper and lower bounds on the objective coefficient of D were both equal to zero. In stage 2 the lower bound is zero and the upper bound is + infinity. This means that any further increase in the expected return on the SP security will not bring about any change in the composition of the portfolio. Therefore, no additional stages involving increasing returns are required, and the process terminates. However, it is still necessary to pursue the consequences of decreasing the return on SP. The maximum permitted decrease in the return on SP is zero. Any slight reduction in its expected return will bring about a different optimal portfolio. Therefore, set its reduction to .001. Calculate new expected returns using the beta coefficients as before. This new formulation is stage 3, and the new expected returns on the securities are .009, .03878, .0488, .008985 and .009015, respectively. The optimal portfolio weights in this problem are .3, .3, .3, .05 and .05, respectively. The new expected portfolio return is 2.987%. The sensitivity report at stage three shows that a new optimal solution will not come about until the SP expected return drops by more than 13.9%. Therefore, an additional reduction of -14% in the expected return on SP would lead to yet a different optimal portfolio. When the new expected returns are calculated with the beta formula the new values all turn out to be negative. The respective expected return coefficients are -1.1310, -.1312, -.1185, -.1333 and -.1288. None of these expected returns is of interest in a long portfolio, so the process is terminated.

CONCLUSION

This work shows how to circumvent a problem in a linear programming formulation of the portfolio selection problem. After the optimal portfolio has been established attention might turn to the question of how the portfolio structure might be altered if any of the expected return parameters of the objective function change. If just one coefficient changes, the standard objective function sensitivity analysis reveals the largest positive or negative change in the expected return on that security that will still result in the optimal portfolio allocations to remain the same. However, the expected returns of the several securities under consideration are quite likely to vary simultaneously and synchronously in response to changing market conditions. It has been shown here how to determine portfolio changes as the expected returns from the several securities under consideration fluctuate. The analysis is limited to the case where all of the fluctuations in expected returns maintain the same ratio to the changes in one of the expected returns.

REFERENCES

- [1] Gennotte, G. and Jung, A. "Investment Strategies Under Transaction Costs: The Finite Horizon Case," *Management Science*, 40 (1994), 385-404.
- [2] Kellerer, H., Mansini, R. and Speranza, M.G., "Selecting Portfolios With Fixed Costs and Minimum Transaction Lots," *Annals of Operations Research*, 99 (2000), 287-304.
- [3] Konno, H. and H. Yamazaki, "Mean-Absolute Deviation Portfolio Optimization Model and Its Application to the Tokyo Stock Market," *Management Science* 37 (1991), 519-531.
- [4] Li, Z., Wang, S.Y. and Deng, X.T., "A Linear Programming Algorithm for Optimal Portfolio Selection with Transaction Costs," *International Journal of System Science*, 31 (2000), 107-117.
- [5] Markowitz, H.M., "Portfolio Selection," *Journal of Finance*, 7 (1952), 77-91.
- [6] Ogryczak, W., "Multiple Criteria Linear Programming Model For Portfolio Selection," *Annals of Operations Research*, 97 (2000), 143-162.
- [7] Schrage, Linus, *LINDO: User's Manual Release 5.3*, Boyd & Fraser Publishing Co., Danvers, MA, 1991.
- [8] Speranza, M.G., "Linear Programming Models for Portfolio Optimization," *Finance* 14 (1993), 107-123.
- [9] Young, M.R., "A Minimax Portfolio Selection Rule With Linear Programming Solution," *Management Science* 44 (1998), 173-183.