THE LONG-TERM PROPERTIES OF A DYNAMIC MARKET

Ferenc Szidarovszky and Abdollah Eskandari, Department of Systems and Industrial Engineering, The University of Arizona, Tucson, AZ 85721-0020, USA, Szidar@sie.arizona.edu
Tamas Zsitvay, Pima College, 1255 N. Stone, Tucson, AZ 85709, USA
Rida Mourtada, Network of Syrian Scientists, P.O. Box 3888, Damascus, Syria

ABSTRACT

The asymptotical properties of dynamic markets with product differentiation are examined with two price adjusting firms. Stability conditions are derived for both best response dynamics and partial adjustments toward best responses. These conditions clearly show that only one of the two firms can stabilize the market regardless of the behavior of the other firm.

INTRODUCTION

The analysis of dynamic markets has a very long history, which can be traced back to the work of Cournot [2]. The most common approach is to consider the manufacturers as the players in a non-cooperative game who face a homogeneous market. The existence and uniqueness of the equilibrium was the first focus of the researchers, and during the past three decades the attention has been turned into analyzing the properties of different dynamic extensions. A comprehensive summary of the earlier studies is given in Okuguchi [3], and applications with multi-product models are presented in Okuguchi and Szidaroszky [4]. The newest trend of research and the newest results are summarized in Bischi et al. [1].

In this paper dynamic markets with two price adjusting firms are examined. After the basic model is obtained the asymptotical stability of best response dynamics is analyzed. Stability condition is derived which is then further generalized for dynamics with partial adjustments toward best responses. The main conclusion of the stability conditions is the fact that any one of the two firms can always stabilize the market regardless of the behavior of the opponent.

THE MATHEMATICAL MODEL AND STABILITY

Consider two firms producing related products. If p_1 and p_2 are the unit prices of the products, then it is assumed that the demand functions are linear:

$$d_i = A_i - B_i p_i + C_i p_j \quad (i=1, 2, j \neq i). \tag{1}$$

For the sake of simplicity linear cost functions are assumed:

$$C_i(d_i) = \alpha_i d_i + \beta_i . \tag{2}$$

Here β_i is the fixed cost and α_i is the marginal cost of firm *i*, and clearly both are positive. In the demand functions A_i and B_i are positive, and C_i is positive or negative if the products are substitutes or complements. The profit of firm *i* is the difference of its revenue and cost:

$$\pi_i = (A_i - B_i p_i + C_i p_j)(p_i - \alpha_i) - \beta_i . \tag{3}$$

Notice that this is a concave parabola in p_i . The best response of firm *i* can be obtained by differentiation,

$$-B_i(p_i-\alpha_i)+A_i-B_ip_i+C_ip_j=0$$

implying that

$$p_i=\frac{1}{2B_i}(B_i\alpha_i+A_i+C_ip_j). \tag{4}$$

If the firms adjust their prices into their best responses at each time period, then this dynamic process can be described by the difference equations

$$p_1(t)=\frac{1}{2B_1}(B_1\alpha_1+A_1+C_1p_2(t-1)) \tag{5}$$

$$p_2(t)=\frac{1}{2B_2}(B_2\alpha_2+A_2+C_2p_1(t-1)). \tag{6}$$

This is a linear system with coefficient matrix

$$\boldsymbol{A_1}=\begin{pmatrix} 0 & \frac{C_1}{2B_1} \\ \frac{C_2}{2B_2} & 0 \end{pmatrix}. \tag{7}$$

It is well known from systems theory [5] that this system is asymptotically stable if and only if the eigenvalues are inside the unit circle. The characteristic polynomial of $\boldsymbol{A_1}$ is quadratic:

$$\varphi_1(\lambda)=\lambda^2-\frac{C_1C_2}{4B_1B_2} \tag{8}$$

implying that system (5)-(6) is asymptotically stable if and only if

$$|C_1C_2|<4B_1B_2. \tag{9}$$

Assume next that the firms adjust their prices adaptively toward best responses, then the dynamic system becomes

$$p_1(t)=p_1(t-1)+s_1\left(\frac{1}{2B_1}(B_1\alpha_1+A_1+C_1p_2(t-1))-p_1(t-1)\right) \tag{10}$$

$$p_2(t)=p_2(t-1)+s_2\left(\frac{1}{2B_2}(B_2\alpha_2+A_2+C_2p_1(t-1))-p_2(t-1)\right) \tag{11}$$

where $0<s_1\le 1$ and $0<s_2\le 1$ are the speeds of adjustment of the firms. This is also a linear system with coefficient matrix

$$\boldsymbol{A_2}=\begin{pmatrix} 1-s_1 & \frac{s_1C_1}{2B_1} \\ \frac{s_2C_2}{2B_2} & 1-s_2 \end{pmatrix}. \tag{12}$$

The characteristic polynomial of this matrix is also quadratic:

$$\varphi_2(\lambda)=(1-s_1-\lambda)(1-s_2-\lambda)-\frac{s_1s_2C_1C_2}{4B_1B_2}$$

$$=\lambda^2+\lambda(s_1+s_2-2)+(1-s_1)(1-s_2)-\frac{s_1s_2C_1C_2}{4B_1B_2}. \tag{13}$$

The eigenvalues are inside the unit circle if and only if

$$(1-s_1)(1-s_2)-\frac{s_1 s_2 C_1 C_2}{4B_1 B_2}<1 \tag{14}$$

and

$$\pm(s_1+s_2-2)+(1-s_1)(1-s_2)-\frac{s_1 s_2 C_1 C_2}{4B_1 B_2}+1>0 \tag{15}$$

Here we use the following result (see for example [1]).

Lemma. Assume p and q are real numbers. The roots of the quadratic equation $\lambda^2+p\lambda+q=0$ are inside the unit circle if and only if

$$q<1$$
$$\pm p+q+1>0\,.$$

Relation (14) can be rewritten as

$$1-s_1-s_2+s_1 s_2(1-\frac{C_1 C_2}{4B_1 B_2})<1, \tag{16}$$

and conditions (15) are equivalent to the following:

$$s_1 s_2(1-\frac{C_1 C_2}{4B_1 B_2})>0 \tag{17}$$

and

$$s_1 s_2(1-\frac{C_1 C_2}{4B_1 B_2})+4-2(s_1+s_2)>0\,. \tag{18}$$

Notice that (17) holds if and only if

$$C_1 C_2<4B_1 B_2, \tag{19}$$

and under this condition (18) also holds, since the speeds of adjustment cannot exceed unity. Relations (16) and (19) show that system (10)-(11) is asymptotically stable if and only if

$$0<1-\frac{C_1 C_2}{4B_1 B_2}<\frac{s_1+s_2}{s_1 s_2}=\frac{1}{s_1}+\frac{1}{s_2}\;. \tag{20}$$

In the special case of $s_1=s_2=1$, system (10)-(11) reduces to (5)-(6), and the stability condition (20) also reduces to (9).

CONCLUSIONS

The stability of price adjusting oligopolies was examined with product differentiation. Sufficient and necessary stability condition was derived (relation (20)). Since the right hand side of (20) decreases as s_1 or/and s_2 increases, it is clear that smaller speeds of adjustments make the dynamic system more stable. The firms have no control on the parameters of the demand functions, and their cost functions are determined by their technologies. They have however full control on the selection of the speeds of adjustment. If $C_1 C_2 \geq 4B_1 B_2$, then the system cannot be asymptotically stable regardless of the values of the speeds of adjustments. Assume that $C_1 C_2<4B_1 B_2$. Then the left part of condition (20) is satisfied. If firm j selects its speed of adjustment so that

$$1-\frac{C_1 C_2}{4B_1 B_2}\leq\frac{1}{s_j}\;, \tag{21}$$

then arbitrary positive value of s_i will stabilize the system. If (21) fails, then firm i is able to stabilize the system by selecting sufficiently small speed of adjustment so that

$$\frac{1}{s_i} > 1 - \frac{C_1 C_2}{4B_1 B_2} - \frac{1}{s_j}. \tag{22}$$

Hence, one firm alone is unable to destabilize the market in the long term, since the other firm always can stabilize it. The Figure 1 shows the stability region with $Q = 1 - \frac{C_1 C_2}{4B_1 B_2}$.

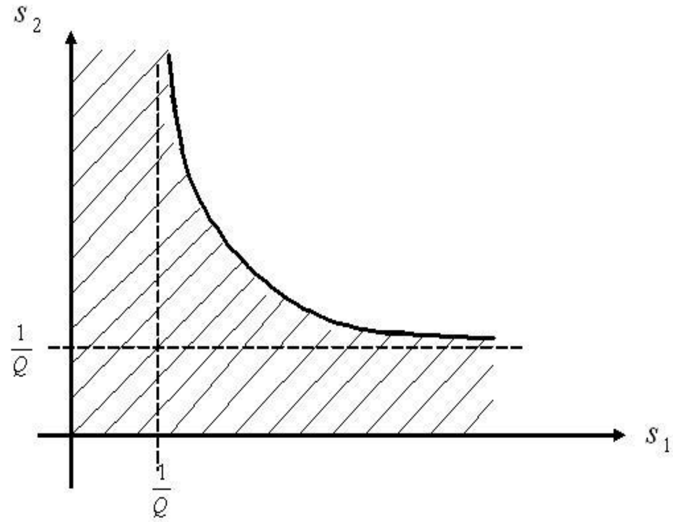

Figure 1. Stability region for the dynamic system

REFERENCES

[1] Bischi, G. I., Chiarella, C., Kopel, M., & Szidarovszky, F. *Nonlinear Oligopolies: Stability and Bifurcations*. Springer-Verlag, Berlin/New York, 2009.

[2] Cournot, A. *Recherches sur les Principes Mathematiques de la Theorie de Richessess*, Hachette, Paris, 1838. (English translation, *Researches into the Mathematical Principles of the Theory of Wealth*, Kelley, New York, 1960.)

[3] Okuguchi, K. *Expectations and Stability in Oligopoly Models*. Springer-Verlag, Berlin/ Heidelberg/New York, 1976.

[4] Okuguchi, K. & Szidarovszky, F. *The Theory of Oligopoly with Multi-Product Firms*. Springer-Verlag, Berlin/Heidelberg/New York (2nd edition), 1999.

[5] Szidarovszky, F. & Bahill, A.T. *Linear Systems Theory*. CRC-Press, Boca Raton/London (2nd edition), 1998.