SOME NOTES ON THE GENERAL PRISONER'S DILEMMA

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ABSTRACT

In this paper, equations are derived for the equilibria of the *N*-person Prisoner's Dilemma game with Pavlovian agents and unequal learning factors. Some simulation results however showed the appearance of additional equilibria. To explain this phenomenon, a refinement of the dynamic equations is given.

Keywords: Game Theory, *N*-person Prisoner's Dilemma, Equilibrium.

INTRODUCTION

Game theory is one of the most frequently applied fields of the science of decision making. It is used to model situations when several decision makers are involved in the process and they have conflicting interests [1]. The simplest game involves two decision makers, who are called the players, and each of them has two alternatives to select from. The well known Prisoner's Dilemma game is one of them. The basic model is the following [2]. Two criminals (called players) committed a crime and became arrested. The district attorney does not have sufficient evidence to convict them with the full crime, only with a less serious charge, unless at least one of them makes a confession. So each of the players has the choice: defecting (D) from the partner by confessing, or cooperating (C) with his partner by remaining silent. They are called the strategies of the players. A version of the payoff table is given below, where the numbers describe the sentences of years in prison. Since the prisoners cannot communicate with each other, there are four possible outcomes: (D,D), (D,C), (C,D) and (C,C). Clearly (D,D) is the only Nash-equilibrium, however if the players would cooperate with their partners, then both of them would get a much better outcome. However, it is dangerous to cooperate, since in the case of defection of the partner, the sentence would become very hash.

1 \ 2	D	C
D	(-5, -5)	(-1, -10)
C	(-10, -1)	(-2, -2)

Table 1: Prisoner's dilemma

In the *N*-person extension of this game, it is defined by two payoff functions, $C(x)$ and $D(x)$, where $C(x)$ is the payoff of the cooperating agents, $D(x)$ is the payoff of defecting agents, and x is the ratio of cooperators in a given neighborhood of the agents [3]. In most cases the neighborhood is defined as the set of all agents. The Nash equilibrium of this game is characterized by the number of cooperators M (and $N - M$ defectors), since the game is symmetric in the defecting and cooperating agents. The case of $M = 0$ (when all agents defect) is an equilibrium if none of the agents want to cooperate. This is the case when

$$D(0) \geq C\left(\frac{1}{N}\right) \quad (1)$$

Similarly, the case of $M = N$ (when all agents cooperate) is an equilibrium if

$$C(1) \geq D\left(1-\frac{1}{N}\right), \quad (2)$$

and an $(M, N - M)$ $(1 \leq M \leq N - 1)$ structure of agents is an equilibrium if

$$C\left(\frac{M}{N}\right) \geq D\left(\frac{M-1}{N}\right) \quad (3)$$

and

$$D\left(\frac{M}{N}\right) \geq C\left(\frac{M+1}{N}\right), \quad (4)$$

which guarantee that both the cooperators and defectors will keep their strategies.

DYNAMIC EXTENSION

In any dynamic extension of static games the behavior of the agents has to be taken into account. The most common assumption is Pavlovian [3], when the agents' strategy changes are random and depend on their payoff values. Let $P_i(t)$ denote the probability that an agent i is cooperating and $x(t)$ the ratio of cooperators at time period of t. Then the probability that an agent will be cooperating in the next time period is given as

$$P_i(t+1) = \begin{cases} P_i(t) + \alpha C(x(t)) & \text{if agent is a cooperator} \\ P_i(t) - \beta D(x(t)) & \text{if agent is a defector} \end{cases} \quad (5)$$

where α and β are the learning factors of cooperating and defecting agents, respectively. By assuming that the agents select strategies independently from each other and taking expectation on both sides, we get the recursive relation

$$x(t+1) = x(t)[x(t) + \alpha C(x(t))] + (1 - x(t))[x(t) - \beta D(x(t))]. \quad (6)$$

The equilibrium of this dynamic process is the steady state, when the state does not change anymore. So it can be obtained as the solution of equation

$$x = x[x + \alpha C(x)] + (1 - x)[x - \beta D(x)]. \quad (7)$$

Consider now the case of linear payoff functions as shown in Figure 1. They are determined by four parameters $S = C(0)$, $P = D(0)$, $R = C(1)$, and $T = D(1)$. Without considering border line cases there are $24 = 4!$ different games. In the case of the Prisoner's Dilemma it is assumed that $T > R > P > S$. Since $D(x) = P + (T - P)x$ and $C(x) = S + (R - S)x$, equation (7) is quadratic, so the maximum number of steady states is two. The right hand side of (7) is the quadratic polynomial

$$R(x) = x(x + \alpha S + \alpha x(R - S)) + (1 - x)(x - \beta P - \beta x(T - P)) = Ax^2 + Bx + C \quad (8)$$

with

$$A = 1 + \alpha(R - S) + \beta(T - P) > 0$$
$$B = \alpha S + \beta P + 1 - \beta T + \beta P = \alpha S + \beta(2P - T) + 1$$
$$C = -\beta P.$$

It is a convex parabola with values $R(0) = -\beta P$ and $R(1) = 1 + \alpha R$. Figure 2 shows the cases of 0, 1, and 2 steady states.

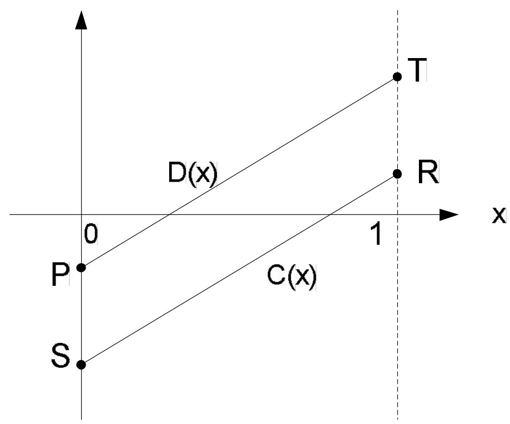

Figure 1: Payoff functions

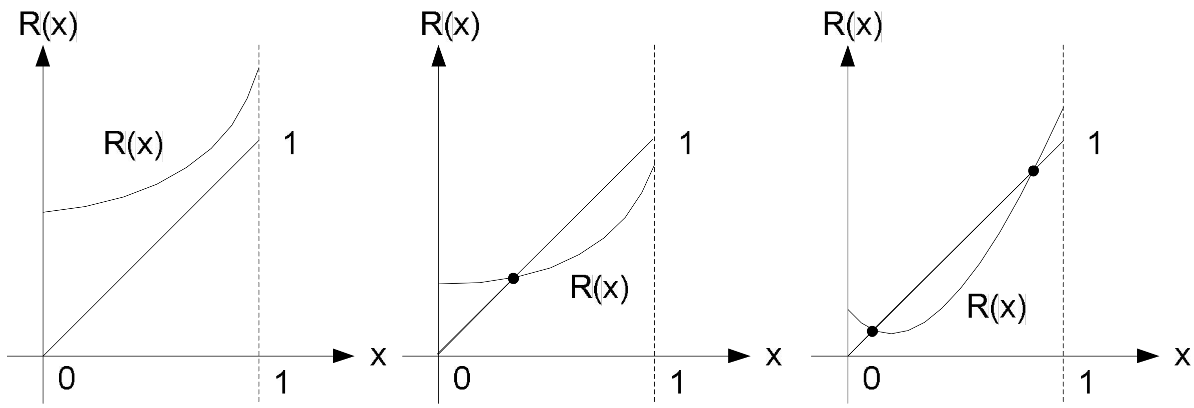

Figure 2: Case of 0, 1, or 2 steady states

In the literature [2] the symmetric case was analyzed, when the learning factors are equal ($\alpha = \beta$). Then the equilibrium equation (7) becomes

$$xC(x) = (1-x)D(x)\,. \tag{9}$$

DISCREPANCY WITH SIMULATION RESULTS

Agent based simulation was used for a large variety of model parameters, when at each time period each agent selected strategy accordingly to equation (5). We always could confirm that the equilibrium obtained by solving equation (7) (as in the symmetric case equation (9)) is the same as the limit point of trajectory $x(t)$ as $t \to \infty$. However in several cases [4] zero and/or unit equilibria was also observed which cannot be explained from these general equations. A more serious re-examination of the dynamic equation gave an answer to this problem. Considering equation (5), if $P_i(t)$ is close to unity and the payoff $C(x(t))$ is large, then the first case might become larger than unity, which has no meaning. The same problem might occur in the second case if $P_i(t)$ is small and $D(x(t))$ is large, so the updated probability value would become negative. Based on this observation the dynamic equation has to be changed as follows:

$$x(t+1) = \begin{cases} R(x(t)) & \text{if } 0 \le R(x(t)) \le 1 \\ 0 & \text{if } R(x(t)) < 0 \\ 1 & \text{if } R(x(t)) > 1 \end{cases} \quad . \tag{10}$$

where *R(x(t))* is the right hand side of equation (6). The emerge of the additional equilibria is illustrated in Figure 3. In the simulation algorithm a similar modification of the updated probability values (5) should be done in order to guarantee that the probability values are between 0 and 1.

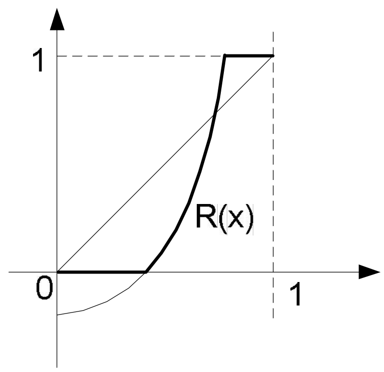

Figure 3: Case of additional equilibria

CONCLUSIONS

In this paper we addressed two related issues. First a general equilibrium equation was derived for the *N*-person prisoner's dilemma by assuming Pavlovian agents and different learning factors. This equation reduces to the well known result known from the literature for the symmetric case. Then the emerge of additional equilibria was examined and we could explain the main reason of this discrepancy between theoretical results and simulation results.

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