POWER-LOG OPTIMIZATION VS. MEAN-VARIANCE OPTIMIZATION FOR DIFFERENT INVESTMENT HORIZONS

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ABSTRACT

This study compares portfolio optimization using Power-Log utility functions with mean-variance optimization for investors with different horizons. Contrary to popular belief it shows that for both short and long horizons, when the downside power for the Power-Log utility function is zero, effectively making it a log utility function, the risk-return characteristics of the optimal portfolio can be far superior to the risk-return characteristics of the mean-variance efficient portfolio with the same expected return. It also shows that the risk-return characteristics of the more conservative optimal Power-Log utility portfolios are superior to those of the corresponding mean-variance efficient portfolios for short horizons.

INTRODUCTION

Investors have different horizons, and the methodology used for constructing an optimal portfolio for an investor should account for that investor's investment horizon. Markowitz (1952) developed the mean-variance framework for portfolio selection and it is the most widely used method for portfolio construction. It is a one-period model. Kelly (1956) and others developed multi-period portfolio theory based on log and power utility functions, where portfolios constructed by using a log utility function will maximize portfolio growth over long horizons. Behavioral finance provides a different perspective on investor actions based on prospect theory, proposed by Kahneman and Tversky (1979) and Tversky and Kahneman (1991, p. 1039), where they define utility separately over gains and losses. Kale (2006) combines the methods of multi-period portfolio theory with some of the tenets of prospect theory to develop Power-Log utility functions that balance growth maximization with downside protection, and shows that optimal Power-Log portfolios have significantly better risk return characteristics than the corresponding mean-variance efficient portfolios when options are included in the asset mix.

The log utility function is a good point of departure for studying portfolio construction methodologies for different horizons, since in theory it produces the best portfolio as the horizon approaches infinity. However, the log utility function has few fans since as Samuelson (1971) stated, "Fortuitously, the utility function log x is the one case that is least complicated to handle when probabilities are intertemporally dependent. This makes log x an attractive candidate for Santa Claus examples in textbooks, but will not endear it to anyone whose psychological tastes deviate significantly from log x." Interestingly, Samuelson goes on to say, "For what it is worth, I may mention that I do not fall into that category," indicating that his own psychological tastes do not deviate significantly from the log utility function. While the log utility function may not represent most investors' risk preferences correctly, it has theoretical merit, so we start our analysis with the log utility function, which happens to be a special case of the Power-Log utility function when the downside power is zero, and then move on to the more realistic Power-Log utility functions with downside powers less than zero.

Most studies about investment horizons have been done in the context of asset allocation, such as Rubinstein (1991), where the asset returns are assumed to have a distribution that is approximately lognormal. To study the effect of asset return distributions that are neither normal, nor lognormal, we have included a call option as one of the assets in the asset mix; its return distribution is distinctly different from normal and lognormal. Its return distribution is highly positively skewed and it has a fat upper tail. The other two assets in the asset mix are cash and stock.

METHODOLOGY

The Markowitz mean-variance methodology for constructing investment portfolios is well known, so we will describe the methodology for constructing optimal portfolios with Power-Log utility functions. Portfolio selection with a utility function uses the expected utility criterion developed by Von Neumann and Morgenstern (1944) and Savage (1964). The following description of the log, power and Power-Log utility functions, the methodology for constructing the joint return distribution, and the resulting optimal portfolios is based on Kale (2006). Each Power-Log utility function is a two-segment utility function, where the utility of gains is modeled with a log utility function and the utility of losses is modeled with a power utility function with power less than or equal to zero. It combines the maximum growth characteristics of the log utility function on the upside, with the scalable downside protection characteristics of the power function on the downside. It is defined as,

$$\begin{aligned} U &= \ln(1+r) \qquad \text{for } r \geq 0 \\ &= \frac{1}{\gamma}(1+r)^{\gamma} \qquad \text{for } r < 0 \end{aligned} \tag{1}$$

where,

r portfolio return

γ power, is less than or equal to 0

Power-Log utility functions conform to the Kahneman and Tversky postulates of reference dependence, loss aversion, and diminishing sensitivity for gains. Investors can vary the level of downside protection they build into their portfolios by changing the downside power. Selecting a downside power of zero is equivalent to using a log utility function for losses, which will result in the construction of the maximum growth portfolio, since the utility function for gains is always a log utility function. Lower values of the downside power represent greater loss aversion since the penalty for losses increases, while the value associated with gains is left unchanged.

Another interesting characteristic of Power-Log utility functions is that they are continuously differentiable across the entire range of returns, which allows the development of fast optimization algorithms for portfolio selection. The algorithm used for this study is a nonlinear mathematical programming algorithm based on an accelerated conjugate direction method developed by Best and Ritter (1976), and has a superlinear rate of convergence.

For a given Power-log utility function the optimal portfolio is selected by maximizing the expected utility of the portfolio. It requires the specification and use of the entire joint distribution of asset returns. As a result all the moments of the distribution of asset returns, including mean, variance, skewness, kurtosis and all the correlations between asset returns are implicitly taken into account.

The three assets we have used for constructing portfolios are, (1) a riskless asset with an annual return of 4%, (2) a risky asset, like a stock index fund, with a mean of annual log return of 10% and standard deviation of annual log return of 20%, and (3) an at-the-money European call option on the risky asset with one year to expiration. Assuming a current price of $100 for the risky asset, the Black-Scholes option pricing model gives us a price of $9.925 for the call option. Using these prices and the probability distribution for the risky asset, we simulated a joint return distribution with 10,000 observations for the three assets. This joint return distribution is used to construct the optimal portfolios with the Power-Log utility functions with different downside powers, and the parameters of this joint return distribution are used to construct the corresponding mean-variance efficient portfolios. Each mean variance efficient portfolio is constructed to have the same expected one-year return as the corresponding Power-Log optimal portfolio.

Once the optimal Power-Log and corresponding mean-variance efficient portfolios have been constructed, we simulate each one's returns for a time horizon of 1, 5, 25 and 100 years. We use 100,000 trials for each simulated return distribution.

OPTIMAL PORTFOLIOS AND HORIZON RETURNS

Table I shows the portfolios constructed by using the Power-Log utility function for downside powers of 0 to –50 in the Optimal Power-Log Portfolios panel, and their expected returns are shown in the second column of the table. The corresponding Markowitz mean-variance (M-V) efficient portfolios shown in the table have been constructed to match the expected one-year return of the optimal Power-Log portfolios.

Table I

		------------ Optimal Power-Log Portfolios ------------				------- M-V Efficient Portfolios -------		
Portfolio	Expected Return (%)	Downside Power	Riskless Weight (%)	Stock Weight (%)	Call Weight (%)	Riskless Weight (%)	Stock Weight (%)	Call Weight (%)
1	21.93	0.00	-58.41	150.62	7.79	-104.97	204.97	0.00
2	17.85	-0.90	1.89	87.99	10.11	-58.36	158.36	0.00
3	15.78	-1.70	27.46	62.11	10.43	-34.59	134.59	0.00
4	12.66	-4.00	58.49	31.84	9.67	1.01	98.99	0.00
5	7.25	-50.00	93.31	1.56	5.13	62.85	37.15	0.00

The optimal Power-Log portfolio in Row 1 of Table I has been constructed with a downside power of 0 and is the maximum growth portfolio, since the Power-Log utility function with a downside power of 0 is the log utility function. It is a very risky, leveraged portfolio with a short position in the riskless asset of 58.41%, but is expected to have the highest growth rate at an infinite horizon. As the downside power is lowered, more downside protection is built into the portfolios and the resulting optimal Power-Log portfolios are progressively more conservative. Remarkably, the weight of the call option relative to the weight of the stock in these portfolios increases as portfolio risk declines. This is due to the significant positive skewness in the call option's returns.

The M-V efficient portfolios shown in Table I have the lowest variance for the given expected returns. All of them have a negligible position in the call option. Clearly the positive skewness of the call's returns carries no weight in the construction of M-V efficient portfolios. The leveraged stock portfolios dominate the call option in mean-variance space. Even though the 0.96 correlation between the stock and call returns is less than 1, the resulting percent weight of the call option in all the M-V efficient portfolios is zero to two decimal places. The M-V efficient portfolio in Row 1 uses a great deal of direct leverage, with a short position in the riskless asset of 104.97%, to achieve an expected return of 21.93%.

For lower levels of expected return, the amount of leverage in the M-V efficient portfolio declines, until it is approximately zero in Row 4. The corresponding Power-Log portfolio in Row 4 is very different; it has a large weight of 58.49% in the riskless asset, a smaller weight of 31.84% in the stock and a significant weight of 9.67% in the call option. The return characteristics of these two portfolios, as well as the other pairs of optimal Power-Log and M-V efficient portfolios are very different as well.
Table II compares the minimum, maximum, standard deviation and negative semideviation below zero for the return distributions of the first pair of optimal Power-Log and M-V efficient portfolios for the four time horizons. The expected returns for the M-V portfolios are 21.97, 14.46, 12.70 and 11.60 percent for the 1, 5, 25 and 100 year horizons respectively. The difference between the two sets of returns increases as the horizon lengthens, since the probability of bankruptcy increases for the M-V efficient portfolio. For virtually every time horizon, the range of returns for each optimal Power-Log portfolio lies above that for the corresponding M-V efficient portfolio, showing that the Power-Log portfolios consistently provide better downside protection and greater upside potential than M-V efficient portfolios. Interestingly, for expected returns above 20%, all the mean-variance efficient portfolios have a minimum return of "–100%," i.e., bankruptcy. Markowitz mean-variance efficiency can be fatal for high-risk high-return portfolios!

Table II

	Expected Ret. (%)	Minimum Return (%)		Maximum Return (%)		Standard Deviation (%)		Neg. Semidev. (%)	
Horizon Years	Power-Log	Power-Log	M-V Efficient	Power-Log	M-V Efficient	Power-Log	M-V Efficient	Power-Log	M-V Efficient
1	21.93	-91.11	-100.00	310.67	282.84	48.54	46.67	17.03	18.38
5	14.54	-53.81	-100.00	120.75	113.52	20.51	21.17	6.27	7.65
25	13.12	-20.05	-100.00	68.10	66.11	9.08	10.96	1.21	5.13
100	12.87	-5.69	-100.00	33.32	33.34	4.52	12.05	0.06	9.88

The standard deviation for the M-V efficient portfolios is slightly lower than the standard deviation for the corresponding Power-Log portfolios, since the mean-variance efficient portfolios have been constructed to minimize the standard deviation for a given expected return. However, standard deviation is an inappropriate measure of risk for assets with asymmetric return distributions. The Power-Log portfolios' larger positive deviations of return from their expected value, when compared to the mean-variance efficient portfolios, appear incorrectly as larger contributions to risk. These larger positive deviations are in fact highly desirable, and should not be penalized. Negative semi-deviation is a better measure of risk. It focuses on downside returns only, and is a measure of downside risk (Markowitz (1959, 1991)). As shown in Table II, the negative semi-deviation below zero for all the Power-Log portfolios is smaller than that for the corresponding M-V efficient portfolios. Based on this measure, the Power-Log portfolios have consistently lower risk than the M-V efficient portfolios, and the difference is dramatic for the 100-year horizon.
Table III compares the asymmetry characteristics of the return distributions of the pair of optimal portfolios for the four time horizons. Skewness, Value at Risk (VaR) at the 99% confidence level, Value to Gain (VtG) at the 99% confidence level (Kale (2006)), and the VtG/VaR ratio all show that the optimal log utility portfolio has less downside risk and better asymmetry characteristics than the corresponding M-V efficient portfolio for all horizons. We do not find that the performance of the optimal log utility portfolio is better than that of the M-V efficient portfolio for very long horizons only as is believed generally; instead we find that the performance of the optimal log utility portfolio is better for all horizons across the board, short, medium and long.

Table III

Horizon Years	Power-Log Expected Return (%)	Skewness		Value at Risk (%)		Value to Gain (%)		VtG / VaR	
		Power-Log	M-V Efficient	Power-Log	M-V Efficient	Power-Log	M-V Efficient	Power-Log	M-V Efficient
1	21.93	0.90	0.61	55.71	66.24	161.08	148.99	2.89	2.25
5	14.54	0.47	0.16	26.13	31.33	69.57	67.50	2.66	2.15
25	13.12	0.20	-2.48	6.61	9.44	35.65	35.59	5.39	3.77
100	12.87	0.10	-7.72	0.00	3.60	23.71	23.84	∞	6.63

Few investors have the stomach for the risk associated with the optimal log utility portfolio. The portfolios in rows 2 through 5 in Table I are progressively more conservative, as the downside power decreases from -0.90 to -50. We will compare the performance of the optimal Power-Log and M-V efficient portfolios in row 3 of Table I. The optimal Power-Log portfolio in row 3 has been constructed using a downside power of -1.70, and it has a one-year expected return of 15.78%. Tables IV and V show the summary statistics for this optimal Power-Log portfolio and the corresponding M-V efficient portfolio.

Table IV

Horizon Years	Power-Log Expected Return (%)	Minimum Return (%)		Maximum Return (%)		Standard Deviation (%)		Neg. Semidev. (%)	
		Power-Log	M-V Efficient	Power-Log	M-V Efficient	Power-Log	M-V Efficient	Power-Log	M-V Efficient
1	15.78	-42.72	-73.75	224.77	187.10	33.45	30.65	0.10	0.10
5	12.27	-24.28	-35.49	87.11	76.95	13.77	13.45	0.03	0.03
25	11.62	-9.85	-11.26	48.63	45.57	6.06	6.02	0.00	0.00
100	11.52	-0.38	-0.97	25.07	25.04	3.01	3.00	0.00	0.00

The expected returns and negative semi-deviation below zero for the two portfolios are virtually the same for each horizon, but the characteristics of their return distributions are quite different. As shown in Table IV, the entire range of returns for the optimal Power-Log portfolio is higher than that for the M-V efficient portfolio at each horizon. Table V also shows that the skewness, VaR, VtG, and the VtG/VaR ratio are all better for the Power-Log portfolio than the M-V efficient portfolio.

Table V

Horizon Years	Power-Log Expected Return (%)	Skewness		Value at Risk (%)		Value to Gain (%)		VtG / VaR	
		Power-Log	M-V Efficient	Power-Log	M-V Efficient	Power-Log	M-V Efficient	Power-Log	M-V Efficient
1	15.78	1.13	0.61	28.13	42.12	115.61	99.20	4.11	2.36
5	12.27	0.59	0.28	13.54	16.13	50.08	46.71	3.70	2.90
25	11.62	0.25	0.11	1.34	1.57	26.85	26.44	20.02	16.86
100	11.52	0.14	0.06	0.00	0.00	18.80	18.93	∞	∞

The remaining optimal Power-Log portfolios in rows 2, 4 and 5 of Table I also follow the same pattern of superior return characteristics when compared to the corresponding M-V efficient portfolios, but the difference between the return distributions of the two sets of optimal portfolios narrows as the horizon lengthens.

CONCLUSION

This study compares portfolio optimization using Power-Log utility functions with mean-variance optimization for investors with different horizons. Contrary to the prevailing wisdom that optimal log utility portfolios have superior performance only when the time horizon is infinite, this study shows that when the downside power for the Power-Log utility function is zero, effectively making it a log utility function, the risk-return characteristics of the optimal portfolio can be far superior to the risk-return characteristics of the mean-variance efficient portfolio with the same expected return for both short and long horizons when an asset with significantly non-normal returns, such as an option, is included in the asset mix. It also shows that the risk-return characteristics of the more conservative portfolios produced with Power-Log utility optimization are superior to the risk-return characteristics of the corresponding mean-variance efficient portfolios for short horizons, and the difference between the portfolios produced by the two optimization methodologies narrows as the horizon lengthens.

REFERENCES

A list of references may be obtained by contacting the authors.