THE OPTIMAL PARAMETERS OF YIELD DISTRIBUTION IN AN EOQ MODEL WITH PLANNED SHORTAGES

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ABSTRACT

This paper extends the previous literature by investigating the effect of investment efforts aimed at improving the production yield, through changing the parameter of yield distribution. The optimal values of policy variables in an EOQ model with planned shortages and random yield are obtained. For the purpose of this paper, yield is defined as the proportion of non-defective items in a production lot. Assuming that the yield probability density function is uniform, the paper presents explicit relationships when the parameter of yield distribution follows a logarithmic investment function.

Keywords: Inventory Theory, Quality

INTRODUCTION

Classical inventory control models ignore the impact of quality on lot size. The reality of stockless production [5] motivated researchers to give careful consideration to the possible relationship between lot size and quality when developing new inventory models. The ultimate goal of stockless production, from an inventory standpoint, is to keep order quantities and production lots to a practical minimum. One important element of a plan aimed at achieving this goal is continuous quality improvement; since the better process quality, the lower the buffer inventory required to meet a specified service level.

Realizing the practical importance of these approaches in operations, researchers have developed a number of inventory models that investigate the relationships among order quantity and quality. Initially, Rosenblatt and Lee [12] investigated the effect of process quality on lot size in the classical economic manufacturing quantity (EMQ) model. Porteus [11] introduced a modified EMQ model that indicates a significant relationship between quality and lot size. In both of these works, the optimal lot size is shown to be smaller than that of the EMQ model. In both the above cases, demand is assumed to be deterministic. Moinzadeh and Lee [6] investigated the effect of defective items on the order quantity and reorder point of a continuous-review inventory model with Poisson demand and constant lead time. Paknejad, Nasri, and Affisco [10] extend this work to consider stochastic demand and constant lead time in the continuous review (s,Q) model. Paknejad, Nasri, and Affisco [9] develop a quality-adjusted EOQ model for the case where both backorders and stockouts are allowed.

Cheng [3] develops a modified Economic Production Quantity (EPQ) model by integrating quality considerations with the lot size. Using classical optimization techniques, the author presents closed form expressions for the optimal lot size and optimal expected fraction acceptable. The optimal lot size is intuitively appealing since it indicates an inverse relationship between lot size and process capability. Goyal et al [4] provides an excellent survey of the early literature on integrating lot size and quality control policies.

In these papers the authors primarily assume that the manufacturer operates a process that is in statistical control. That is, the production yield, defined as proportion of non-defective items in each lot, is known and constant. Such an assumption induces a situation where the number of non-defective units in each lot follows a binomial distribution. This assumption is also made in Affisco, Paknejad, and Nasri [1] for the case of the EOQ and Affisco, Paknejad, and Nasri [2] for the case of the joint economic lot size model.

In a recent paper, Nasri, Paknejad, and Affisco [8] study the relationship between order quantity and quality for processes that have not yet achieved the state of statistical control, such as those that are in the initial stages of implementing a quality program. Specifically, the authors assume that each lot contains a random proportion of defective units, which implies that the production yield is also random. Upon arrival, the purchaser inspects the entire lot. They further assume that the purchaser's inspection process is perfect, and that all rejected items are returned to the vendor at no cost to the purchaser. In addition, it is assumed that the inspection cost is paid by the vendor. Based on this scenario, the authors in [8] adjusted the EOQ with planned shortages model for the quality factor. In addition to the general relationships, the authors provided closed form expressions for two special cases where the proportion of defective items follows uniform or exponential distributions. In a more recent paper, Nasri, Paknejad, and Affisco [7] extended the results to the case of the Economic Manufacturing Quantity (EMQ) with backorders model.

The results developed in Nasri, Paknejad, and Affisco [7, 8] are significant and surely consistent with the wide-spread understanding that yield is a major concern for all manufacturing organizations. However, these models ignore the fact that these organizations routinely consider investing in yield improvement programs and implicitly assume that the parameters of yield distributions are known and constant, uncontrollable by management.

The main objective of this paper is to extend the previous work and study the impact of efforts devoted to yield improvement programs on lot sizes and backorder levels. Such efforts include those directed at process improvement (including process redesign), total quality maintenance, the more effective use of information technology in process management, modernizing process controls, implementing statistical process control, and others. It is reasonable to assume that these efforts ultimately alter the parameters of the yield distributions. The paper attempts to achieve its objective by considering the parameters of yield distribution (defined as yield parameters) as decision variables, rather than as known constants, and develops a mathematical model which captures the relationship between investment costs aimed at changing yield parameters and other inventory related costs identified in Nasri, Paknejad, and Affisco [8]. The paper presents the optimal values of decision variables in closed forms for the specific case of uniform probability density function for yield distribution and logarithmic investment function for changing the shape of yield distribution.

2. MODEL AND ASSUMPTIONS

Consider the traditional Economic Order Quantity (EOQ) model that allows shortages and backordering with the following total annual cost function

$$C_{Trad}(Q,S) = \frac{D}{Q}K + \frac{(Q-S)^2}{2Q}C_h + \frac{S^2}{2Q}C_b \tag{1}$$

where

D = Annual Demand in units, Q = Lot size per order,

S = Number of units backordered, K = Ordering cost per order,

c_h= Holding cost per unit per year, c_b= Backordering cost per unit per year.

The results of classical optimization yields the following well-known expressions for the optimal values for the lot size, Q^*_{Trad} , units backordered, S^*_{Trad} , and the annual cost, C^*_{Trad}

$$Q^*_{Trad} = \sqrt{\frac{2DK}{c_h}\left(\frac{c_h + c_b}{c_b}\right)} \tag{2}$$

$$S^*_{Trad} = \sqrt{\frac{2DK}{c_b}\left(\frac{c_h}{c_h + c_b}\right)} \tag{3}$$

and

$$C^*_{Trad} = \sqrt{2DK\,c_h\left(\frac{c_b}{c_h + c_b}\right)} \tag{4}$$

Implicit in these derivations is that all units produced by the vendor, in response to the purchaser's order, are of acceptable quality. Now, assume that this is not the case. Specifically, assume that each lot contains a random proportion of defective units. Upon arrival, the purchaser inspects the entire lot. Further assume that the purchaser's inspection process is perfect, and that all rejected items are returned to the vendor at no cost to the purchaser. In addition, the inspection cost is paid by the vendor. Of course, it is likely that the vendor will recover some of these costs from the purchaser either directly or indirectly. Based on this scenario, we adjust the Basic EOQ with planned shortages model for the quality factor. Let

= Yield, being defined as the proportion of non-defective items in an order lot, [0, 1], a continuous random variable,

f( ) = Probability density function of ,

E( ) = First moment of ,

E(^{2}) = Second moment of ,

y = Q = Number of non-defective items in a lot,

c(y) = Total cost per cycle given that there are y non-defective items in a lot of size Q,

T = y/D = Cycle time, time between two successive placement of orders,

E(.) = Mathematical expectation,

The total cost per cycle is

$$c(y) = K + \frac{(y-S)^2}{2D}c_h + \frac{S^2}{2D}c_b = K + \frac{(\lambda Q - S)^2}{2D}c_h + \frac{S^2}{2D}c_b \quad . \tag{5}$$

The average cycle time and cycle cost are

$$E(T) = \frac{E(y)}{D} = \frac{E(\lambda Q)}{D} = \frac{Q}{D}\,E(\lambda) \quad , \tag{6}$$

and

$$E(c) = K + \frac{c_h.Q}{2D}\left[Q.E(\lambda^2) - 2S.E(\lambda)\right] + \frac{c_h + c_b}{2D}.S^2 \quad . \tag{7}$$

The expected total annual cost is

$$EAC_{adj}(Q,S) = \frac{DK}{E(\lambda)Q} + \left[\frac{E(\lambda^2)Q}{2E(\lambda)} - S\right]c_h + \frac{(c_h + c_b)S^2}{2E(\lambda)Q} \quad . \tag{8}$$

In what follows we assume that the probability density function of λ is uniform with location parameter C and scale parameter C^c = 1-C. That is,

$$f(\lambda) = \begin{cases} \dfrac{1}{1-C} & for \ \ \mathrm{C} \le \lambda \le 1, \text{ where } 0 \le \mathrm{C}<1 \\ 0 & \text{otherwise} \end{cases} . \tag{9}$$

In this case

$$E(\lambda) = \frac{1+C}{2} \quad , \tag{10}$$

and

$$E(\lambda^2) = \frac{1+C+C^2}{3} \quad . \tag{11}$$

Substituting (9), (10), and (11) into (8) and using calculus, the optimal values for the order quantity, Q^*_{adj} , units backordered, S^*_{adj} , and expected total annual cost, $EAC^*_{adj}(S,Q)$, are found as follows

$$Q^*_{adj,u} = \left(\frac{2}{1+C}\right)\sqrt{\frac{2DK}{c_h\left[\left(\frac{1}{3}\right)\left(\frac{1-C}{1+C}\right)^2 + \left(\frac{c_b}{c_h+c_b}\right)\right]}}, \tag{12}$$

$$S^*_{adj,u} = \sqrt{\frac{2DK\left(\dfrac{c_h}{c_h + c_b}\right)}{c_b\left[\left(\frac{1}{3}\right)\left(\frac{1-C}{1+C}\right)^2\left(\frac{c_h+c_b}{c_b}\right)+1\right]}}, \tag{13}$$

and

$$EAC^*_{adj,u} = \sqrt{2DKc_h\left[\left(\frac{1}{3}\right)\left(\frac{1-C}{1+C}\right)^2 + \frac{c_b}{c_h+c_b}\right]} \quad . \tag{14}$$

Please note that in (12) through (14), if the yield location parameter C = 1, then the yield scale parameter C^c = 0. In such case, the quality is perfect and the quality-adjusted model of this paper simply reduces to the traditional EOQ with planned shortages model expressed in (2) through (4).

3. THE OPTIMAL YIELD PARAMETER MODEL

The decision variables in the model of previous section are Q and S for a fixed location parameter of yield distribution, C. The value of this parameter determines the values of both mean and variance of yield distribution. As C approaches one, yield rate increases and yield variability diminishes, hence quality improves. In this paper, we assume that the option of investing to increase C is available. To evaluate the economic trade-offs associated with this investment option, we introduce a companion yield parameter, Θ , as follows:

$$\Theta = \frac{1-C}{1+C} \quad \text{for } \ 0 < \Theta \le 1 \quad , \tag{15}$$

Please note that as C increases from 0 to 1, Θ decreases from 1 to zero. Thus, reducing Θ implies increasing C and, hence, improving quality.
Now, we consider Θ to be a decision variable and seek to minimize the average annual cost composed of, investment cost of reducing yield parameter to a new level, ordering, shortage, and holding costs. Specifically, we seek to minimize

$$C(Q,S,\Theta)=i.a_{\Theta}(\Theta)+EAC^{*}_{adj,u}(Q,S) \ , \tag{16}$$

Subject to

$$0<\Theta\leq 1 \quad , \tag{17}$$

where i is the cost of capital, $a_{\Theta}(\Theta)$ is a convex and strictly decreasing function of Θ representing the cost of reducing the yield parameter to the level Θ, $EAC_{adj,u}(Q,S)$ is the sum of all inventory related costs given in equation (8) for the case of uniform yield distribution, and Θ_0 is the original quality parameter.
One reasonable way of dealing with this optimization problem is to use a sequential approach, suggested by Porteus [11]. In this case, we hold Θ fixed, optimize over Q and S to obtain $Q^{*}_{adj}(\Theta)$ and $S^{*}_{adj}(\Theta)$, given by equations (12) and (13) with C = (1-Θ)/(1+Θ), and then optimize over Θ. That is, we seek to minimize

$$w(\Theta)=i.a_{\Theta}(\Theta)+EAC^{*}_{adj}(\Theta) \ , \tag{18}$$

where $EAC^{*}_{adj}(\Theta)$ is given by equation (14) modified for Θ as follows:

$$EAC^{*}_{adj}(\Theta) = \sqrt{2DKc_h\left[\frac{1}{3}\Theta^2+\frac{c_b}{c_h+c_b}\right]}. \tag{19}$$

Of course if the optimal Θ obtained in this way does not satisfy restriction (17), we should not make any investment and the results of the quality adjusted EOQ model with planned shortages hold. Please note that it may not always be possible to carry out the above minimization except for some special cases of $a_{\Theta}(\Theta)$. The following section considers the case of logarithmic investment function.

4. THE LOGARITHMIC INVESTMENT FUNCTION

This particular function is used in previous research [7, 8, 9, 10, 11] dealing with quality improvement as well as setup cost reduction. Its use in this paper is based on the idea that yield improvement should exhibit decreasing marginal return. In this case the yield parameter, Θ, declines exponentially as the investment amount, a_{Θ}, is increased. That is

$$a_{\Theta}(\Theta)=\frac{1}{\Gamma}Ln\frac{\Theta_0}{\Theta} \qquad for \qquad 0<\Theta\leq\Theta_0 \ , \tag{20}$$

where Γ is the percentage decrease in Θ per dollar increase in $a_{\Theta}\Theta$. Here our main objective is to minimize $w(\Theta)$ after substituting (19) and (20) into (18).
Theorem: If Θ_0 and Γ are strictly positive, then the following hold:
a) The optimal value of the yield parameter is given by

$$\Theta^{**}=\min\left\{\Theta_0,\Theta_{imp}\right\}, \tag{21}$$

where Θ_0 = the original yield parameter and

$$\Theta_{imp} = \left\{ \frac{3i^2}{4\Gamma^2 DKc_h} \left[1 + \sqrt{1 + \frac{8\Gamma^2 DKc_h}{i^2} \left(\frac{c_b}{c_h + c_b} \right)} \right] \right\}^{\frac{1}{2}}, \tag{22}$$

(b) The resulting optimal yield location and optimal yield scale parameters are

$$C^{**} = \max\left\{C_0, C_{imp}\right\}, \tag{23}$$

$$C^{c**} = \min(C_0^c, C_{imp}^c), \tag{24}$$

where C_0 and C_0^c are the original location and scale parameters, and

$$C_{imp} = \frac{1-\Theta_{imp}}{1+\Theta_{imp}}, \tag{25}$$

$$C_{imp}^c = \frac{2\Theta_{imp}}{1+\Theta_{imp}}. \tag{26}$$

(c) The optimal values for the order quantity, Q_{adj}^{**}, units backordered, S_{adj}^{**}, and expected total annual cost, EAC_{adj}^{**}, are as follows

$$Q_{adj}^{**} = \begin{cases} Q_{adj}^{*} & if\ \Theta_{imp} \geq \Theta_0 \\ Q_{imp} & if\ \Theta_{imp} < \Theta_0 \end{cases}, \tag{27}$$

$$S_{adj}^{**} = \min(S_{adj}^{*}, S_{imp}) = \begin{cases} S_{adj}^{*} & if\ \Theta_{imp} \geq \Theta_0 \\ S_{imp} & if\ \Theta_{imp} < \Theta_0 \end{cases}, \tag{28}$$

$$EAC_{adj}^{**} = \min(EAC_{adj}^{*}, EAC_{imp}) = \begin{cases} EAC_{adj}^{*} & if\ \Theta_{imp} \geq \Theta_0 \\ EAC_{imp} & if\ \Theta_{imp} < \Theta_0 \end{cases}, \tag{29}$$

where Q_{adj}^{*}, S_{adj}^{*}, and EAC_{adj}^{*} are given by (12), (13), (14), and

$$Q_{imp} = \left(1+\Theta_{imp}\right) \sqrt{\frac{2DK}{c_h\left[\left(\frac{1}{3}\right)\Theta_{imp}^2 + \left(\frac{c_b}{c_h+c_b}\right)\right]}}, \tag{30}$$

$$S_{imp} = \sqrt{\frac{2DK\left(\frac{c_h}{c_h + c_b}\right)}{c_b\left[\left(\frac{1}{3}\right)\Theta_{imp}^2\left(\frac{c_h+c_b}{c_b}\right)+1\right]}}, \tag{31}$$

and

$$EAC_{imp} = \sqrt{2DKc_h\left[\left(\frac{1}{3}\right)\Theta_{imp}^2 + \frac{c_b}{c_h+c_b}\right]} \quad . \tag{32}$$

It is interesting to note that Θ_{imp}, C_{imp}, C_{imp}^c, Q_{imp}, S_{imp}, and EAC_{imp} do not depend on Θ_0. It should also be noted that when $\Theta_{imp} \geq \Theta_0$, we should not make any investment. In this case Θ_0 will be used in place of Θ_{imp} and equations (30), (31), and (32) will be replaced by equations (12) through (14). Further, when quality is perfect (i.e., $\Theta=0$, $C=1$, and $C^c=0$), then the results of this paper simply reduce to the corresponding results of the traditional EOQ with planned shortages model given in equations (2) through (4). Details of the proofs are omitted.

5. CONCLUSION

This paper extended a recently developed quality-adjusted EOQ with planned shortages model with random production yield [8]. The paper studied the economic trade-offs associated with investment efforts aimed at improving the production yield through changes in both the location parameter as well as the scale parameter of yield distribution. Assuming that the yield probability density function is uniform, the paper presented explicit relationships for the optimal values of location and scale parameters, order quantity, backorder level, and expected total annual cost for the case of logarithmic investment function.

A full set of references is available upon request from the authors.

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