ON OPTIMAL AGE REPLACEMENT DECISION OF NON-REPAIRABLE SYSTEMS

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ABSTRACT

The classical model for finding optimal preventive replacement policies is revisited. In minimizing expected cost per unit time in a cycle, we first introduce discount factors since the present value of the replacement cost depends on the time when replacement actually occurs. Then certainty equivalent is used to take risk into account arising from the randomness of the cost per unit time, since the time of failure is uncertain. A simple numerical example with Weibull variable illustrates the methodology.

INTRODUCTION

The problem of optimal age replacement is one of the most important issues in reliability engineering. Several alternative methods were introduced and suggested in the literature (for example, Elsayed, 1996; Jardine, 2006; Coolen-Schrijner and Coolen, 2004). They can be divided into two major groups. Methods based on the renewal theory assume that replacement costs do not change in time, and the replacement is done with an identical item all the times. In applying one-cycle criterion, the expected costs per unit time for a single cycle are minimized instead of a long-term expectation. In both cases, only the expectation is considered and no attention is given to the risk arising from the randomness of the objective functions. In the mathematical economics literature, random outcomes are usually replaced by their certainty equivalents transforming the stochastic problem to a deterministic programming approach (Sargent, 1979).

In this paper, a simple single cycle model is extended in two directions. First, we will include discount factors in the cost occurring by replacements and then the certainty equivalent of the resulting objective function will be introduced. A simple numerical example will illustrate the methodology.

THE MATHEMATICAL MODEL

Let X denote the time to failure of a system with pdf $f(X)$, CDF $F(X)$, reliability function $R(X) = 1 - F(X)$ and failure rate $\rho(X) = f(X) / R(X)$. Let C_p and C_f be the preventive replacement cost and the cost of replacing the failed item with $C_f > C_p$.

Assume that our plan is to replace the item after T time unless it fails before, in which case it is immediately replaced. So, the cycle length is $min\{T;X\}$. The cost per unit time in a cycle is clearly

$$CPUT = \begin{cases} \frac{C_f}{X} \text{ if } X \le T \\ \\ \frac{C_p}{T} \text{ if } X > T \end{cases} \tag{1}$$

with expectation

$$\begin{aligned} g_1(T) &= \int_0^T \frac{C_f}{X} f(X) dX + \int_T^\infty \frac{C_p}{T} f(X) dX \\ &= C_f \int_0^T \frac{f(X)}{X} dX + \frac{C_p}{T} R(T) \end{aligned} \tag{2}$$

which is minimized in the classical model. If the item has a long lifetime, then the present values of the costs occurring in later times have to be considered. If r is the discount factor, then the present value of the cost per unit time becomes

$$PVCPUT = \begin{cases} \frac{C_f}{X} e^{-rX} \text{ if } X \le T \\ \\ \frac{C_p}{T} e^{-rT} \text{ if } X > T \end{cases} \tag{3}$$

with expected value

$$g_2(T) = C_f \int_0^T \frac{f(X)}{X} e^{-rX} dX + \frac{C_p}{T} R(T) e^{-rT} \tag{4}$$

By differentiation

$$g_2'(T) = \frac{(C_f - C_p) e^{-rT} R(T)}{T} \left(\rho(T) - \frac{C_p\,(1+rT)}{T\,(C_f - C_p)} \right) \tag{5}$$

In most cases, $\rho(T)$ strictly increases in T, $\rho(0) = 0$ and $lim_{T\to\infty}\, \rho(T)$ is either infinity (e.g. for Weibull distribution) or finite (e.g. for gamma distribution). The last term of (5) can be rewritten as

$$\frac{C_p}{T\,(C_f - C_p)} + \frac{C_p\, r}{C_f - C_p}\,, \tag{6}$$

which is a strictly decreasing function in T, its limit at zero is infinity and its limit at infinity is $C_p r \,/\, (C_f - C_p)$. Therefore, there is a unique solution if

$$lim_{T\to\infty}\, \rho(T) > \frac{C_p r}{C_f - C_p}\,, \tag{7}$$

and it gives the global minimum of the objective function, otherwise $g_2(T)$ strictly decreases in T implying that there is no finite optimum. In this case, no preventive replacement is needed.

In order to find the certainty equivalent of the random cost (3), we need to determine its second moment:

$$ {C_f}^2 \int_0^T \frac{e^{-2rX}}{X^2} f(X) dX + \frac{{C_p}^2 e^{-2rT}}{T^2} R(T) \;, \tag{8} $$

so its variance is

$$ VAR(T) = {C_f}^2 \int_0^T \frac{e^{-2rX}}{X^2} f(X) dX + \frac{{C_p}^2 e^{-2rT}}{T^2} R(T) - g_2(T)^2 \tag{9} $$

The certainty equivalent is a linear combination of the expectation and variance,

$$ g_3(T) = \alpha \, g_2(T) + (1-\alpha) \, VAR(T) \;, \tag{10} $$

which is minimized. It is assumed that $0 \le \alpha \le 1$. If no consideration is given to the risk, then only the expectation is minimized and so $\alpha = 1$ is selected. If only risk is considered, then $\alpha = 0$ is the choice. The ratio $(1 - \alpha) / \alpha$ shows the risk seeking attitude of the decision maker.

NUMERICAL EXAMPLE

Assume that the time to failure follows a Weibull distribution with parameters $\beta = 2.5$ and $\eta = 5$. Furthermore, the cost data are $C_p = 500$ and $C_f = 600$ dollars. The discount factor is assumed to be 5%, that is, $r = 0.05$. The classical model is a special case of (3)-(5) with $r = 0$, so the optimal preventive replacement time is the solution of equation

$$ \frac{\beta}{\eta} \left(\frac{T}{\eta}\right)^{\beta-1} - \frac{C_p}{T \, (C_f - C_p)} = 0 \;, \tag{11} $$

which gives a closed form optimum

$$ T_1^* = \eta \left(\frac{C_p}{\beta \, (C_f - C_p)}\right)^{1/\beta} \approx 6.60 \tag{12} $$

If the discount rate is included, then the optimum is obtained by solving the single variable equation

$$ \frac{\beta}{\eta} \left(\frac{T}{\eta}\right)^{\beta-1} - \frac{C_p \, (1+rT)}{T \, (C_f - C_p)} = 0 \;, \tag{13} $$

where the left hand side strictly increases in T, having negative infinity limit at zero and positive infinity limit as T tends to infinity. This monotonic equation can be easily solved by using standard methods (see for example, Yakowitz and Szidarovszky, 1989). The actual solution is $T_2^* \approx 7.49$.

In minimizing the certainty equivalent the first order condition becomes very complicated, so a simple one-dimensional search algorithm can be used. With $\alpha = 1$, only $g_2(T)$ is minimized, and with different values of α, the optimum has changed as shown in the following table:

α	1	0.999	0.99	0.95	0.909
T_3^*	7.49	7.31	4.15	0.306	0.263

Table 1. Optimal preventive replacement times.

The values of $g_2(T)$ show a convex curve with a unique minimum at $T_1^* = 7.49$, however *VAR(T)* is a strictly increasing function with much larger values than $g_2(T)$. Therefore, as its weight $(1 - \alpha)$ becomes larger, this increasing function becomes more and more dominating, which results in smaller and smaller minimizing T values. As $\alpha \rightarrow 0$, the optimum T converges to zero.

CONCLUSIONS

Two extensions of a classical model of determining the time of preventive replacement of non-repairable systems is introduced. In the first case, we introduced discount factors in the costs of replacements. In the second case, certainty equivalent was used in order to take risks into account. A simple numerical example with Weibull time-to-failure distribution illustrated the methodology.

The numerical results suggest two observations. The first is that the planned replacement time increases if discount factors are considered. This is clearly the case with strictly increasing failure rates, since with any $r > 0$ expression (6) is larger than that with $r = 0$. If certainty equivalent is considered, then with decreasing value of α (which means larger importance to the risk), the optimum time of replacement decreases.

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