A Gauge Repeatability and Reproducibility Study for Multivariate Measurement Systems

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Abstract

In this paper, the correlation coefficients among tolerances are taken into account to revise the multivariate precision-to-tolerance (P/T) ratio as proposed by Majeske (2008). We compare the performance of our revised P/T ratio with that of the existing ratios. The simulation results show that our revised P/T ratio outperforms the existing ones. Furthermore, the optimal allocation of *por* parameters is discussed using the shortest confidence interval of the revised P/T ratio. Finally, a numerical example is given to illustrate the appropriateness of our proposed P/T ratio. A reference table with optimal allocations of parameters is also provided. Hopefully, it can be served as a useful guideline for quality practitioners when conducting multivariate GRR study in industries.

Keywords: multivariate measurement system analysis; multivariate gauge repeatability and reproducibility; precision-to-tolerance ratio.

Introduction

Measurement system analysis (MSA) plays an important role in helping organizations to improve their product quality. Generally speaking, the gauge repeatability and reproducibility (GRR) study is performed according to the MSA handbook. Usually, GRR study for assessing the adequacy of gauge variation needs to be conducted prior to the process capability analysis. Good quality products can only be achieved through an adequate measurement system. Hence, finding ways to ensure the quality of a measurement system becomes an important task for quality practitioners. Moreover, in performing the GRR study, most industries today are using the approval criteria of Precision to Tolerance (P/T) ratio as stipulated in QS9000. Quality practitioners are also expected to determine the optimal allocation of sample size of parts (p), number of operators (o) and repeated measurements (r) for economic reasons. As the total number of measurements (n) increase, the estimated total variation becomes more precise, but the related inspection time and costs will be increased as well. Traditional MSA only considers a single quality characteristic. With the advent of modern technology, industrial products have become very sophisticated with more than one quality characteristic. Thus, it becomes necessary to perform multivariate GRR (MGRR) analysis for a measurement system when collecting data with multiple responses. Recently, principle component analysis (PCA) and MANOVA have been proposed for analyzing gauge variation by Wang and Yang [12] and Majeske [4] respectively. However, neither the correlation coefficients among tolerances nor the optimal allocation of *por* parameters are discussed in performing their MGRR studies. Therefore, this paper aims to develop a new P/T ratio with the consideration of optimal allocation of por parameters when conducting the MGRR study.

Literature review for MGRR

Wang and Yang [12] applied principal component analysis (PCA) method to transform multiple characteristics into one or a few irrelevant variables. Then, these irrelevant variables are analyzed using analysis of variance. Majeske [4] applied multivariate analysis variance method (MANOVA) to estimate the variance-covariance matrices for a multivariate measurement system. He considered the two-way random-effects MANOVA model with the interaction term

$$\mathbf{X} = \boldsymbol{\mu} + \boldsymbol{P}_i + \boldsymbol{O}_j + (\boldsymbol{PO})_{ij} + \boldsymbol{\varepsilon}_{ijk} \quad \begin{cases} i = 1, \dots, p \\ j = 1, \dots, o \\ k = 1, \dots, r \end{cases} \qquad (1)$$

He further proposed a multivariate precision-to-tolerance ratio as

$$P/T = \left[\frac{ \dfrac{ \left(\prod_{l=1}^{v} \sqrt{\chi^2_{99.73,v} \lambda_{G_l}} \right) \pi^{\frac{v}{2}} }{ \Gamma\left(1+\frac{v}{2}\right) } }{ \dfrac{ \left(\prod_{l=1}^{v} \frac{TOL_l}{2} \right) \pi^{\frac{v}{2}} }{ \Gamma\left(1+\frac{v}{2}\right) } } \right]^{\frac{1}{v}} = \left(\prod_{l=1}^{v} \frac{2\sqrt{\chi^2_{99.73,v} \lambda_{G_l}}}{TOL_l} \right)^{\frac{1}{v}} \qquad (2)$$

where TOL_l denotes the specification tolerance for the *l*th quality characteristic and $\chi^2_{1-\alpha,v}$ denotes the (1−α)th percentile of a chi-square distribution with *v* degrees of freedom. Other multivariate measurement system analysis methods using PCA related methods can be referred to in Osma [7]、Peruchi *et al.* [10] and Wang [11].

Developing Revised P/T ratio In this paper, we consider the two-way random-effects MANOVA model with the interaction term as shown in equation (1). According to equation (2), the numerator in P/T ratio can be rewritten as

$$\frac{|\boldsymbol{\Sigma}_G|^{\frac{1}{2}} \left(\pi \chi^2_{99.73,v}\right)^{\frac{v}{2}}}{\Gamma\left(1+\frac{v}{2}\right)}$$

where $\boldsymbol{\Sigma}_G$ is the variance-covariance matrix of gauge error. To take the correlation among multiple quality characteristics into account for MGRR, we adopted the idea proposed by Pan and Lee [9] and revised the denominator of the P/T ratio proposed by Majeske [4]. Accordingly, the revised P/T ratio is defined as

$$\mathrm{P/T_R} = \left[\frac{|\boldsymbol{\Sigma}_G|^{\frac{1}{2}}(\pi\chi^2_{99.73,v})^{\frac{v}{2}}\left(\Gamma\left(\frac{v}{2}+1\right)\right)^{-1}}{|\boldsymbol{A}^*|^{\frac{1}{2}}(\pi\chi^2_{99.73,v})^{\frac{v}{2}}\left(\Gamma\left(\frac{v}{2}+1\right)\right)^{-1}}\right]^{\frac{1}{v}} = \left[\frac{|\boldsymbol{\Sigma}_G|}{|\boldsymbol{A}^*|}\right]^{\frac{1}{2v}} \qquad (3)$$

where the elements of matrix $\mathbf{A}^*$ are given in Pan and Lee [9]. It is worthy to note that both of the matrices $\boldsymbol{\Sigma}_G$ and $\mathbf{A}^*$ are positive definite.

Comparing the simulation results for $\mathbf{P/T_R}$ and other P/T ratios

In this section, various simulation studies are conducted to compare the performance of the revised P/T ratio ($\mathbf{P/T_R}$) with that of the existing ones. Two scenarios are considered here: random-effects MANOVA models without and with interaction term. The MANOVA model with interaction term is further explained below.

Let $\mathbf{X}_{\mathrm{ijk}}$ denote the measurement by operator *j* on part *i* at replication *k*, then the random-effects MANOVA model with interaction can be written as

$$\mathbf{X}_{\mathrm{ijk}} = \boldsymbol{\mu} + \mathbf{P}_i + \mathbf{O}_j + (\mathbf{PO})_{\mathrm{ij}} + \mathbf{E}_{\mathrm{ijk}} \qquad (4)$$

where $\boldsymbol{\mu}$ is the total mean vector, $\mathbf{P}_i$ is the effect of the *i*th part, $\mathbf{O}_j$ is the effect of the jth operator, $(\mathbf{PO})_{\mathrm{ij}}$ is the effect of part-operator interaction and $\mathbf{E}_{\mathrm{ijk}}$ is the random error representing the repeatability. All of the above four effects are assumed to be random effects that are multivariate normally distributed with mean vector of zero and variance-covariance matrix of $\boldsymbol{\Sigma}_{\mathrm{P}}$, $\boldsymbol{\Sigma}_{\mathrm{O}}, \boldsymbol{\Sigma}_{PO}$ and $\boldsymbol{\Sigma}_{\mathrm{E}}$, respectively. Without loss of generality, in this simulation study, we assume that a MGRR study with three quality characteristics is conducted. The settings of the variance-covariance matrix for $\boldsymbol{\Sigma}_{\mathrm{P}}$, $\boldsymbol{\Sigma}_{\mathrm{O}}$ and $\boldsymbol{\Sigma}_E$ are the same as the previous section, while the variance-covariance matrix for interaction is set as

$$\boldsymbol{\Sigma}_{PO} = \begin{bmatrix} 1 & \rho_{po_{12}} & \rho_{po_{13}} \\ \rho_{po_{12}} & 1 & \rho_{po_{23}} \\ \rho_{po_{13}} & \rho_{po_{23}} & 1 \end{bmatrix}$$

where$\rho_{po_{12}}$, $\rho_{po_{13}}$, and $\rho_{po_{23}}$ are generated from a uniform distribution with the interval (0, 1). The simulated data is generated given that $p = 25$, $o = 5$, $r = 3$ and the average values of $\widehat{\mathrm{P/T}}$ are computed based on 1000 simulation runs. Based on the simulation results shown in Figure 1, one can find that the biases of Majeske [4]'s and Wang and Yang [12]'s P/T ratios are deviated from zero. In contrast, the biases of our revised P/T ratio are closer to zero.

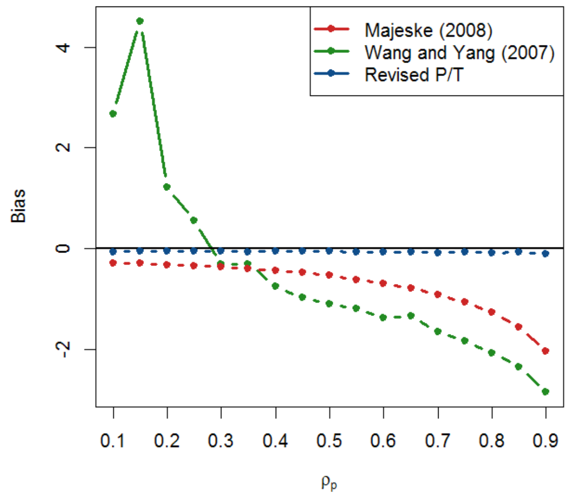

FIGURE 1. Comparison of the simulation results of bias for various P/T ratios under different product correlation coefficient ρ_p (considering the PO interaction)

In order to determine the optimal allocation of various combinations among sample size of parts (p), number of operators (o) and replications (r), different widths of confidence intervals for our proposed $\mathrm{P/T_R}$ based on the MANOVA random effect model are calculated. First, we derive the confidence interval for $\mathrm{P/T_R}$. According to equation (3), the estimator of P/T_R is given by

$$\widehat{P/T_R} = \left[\frac{|\mathbf{S}_G|}{|\mathbf{A}^*|}\right]^{\frac{1}{2v}}$$

where $\mathbf{S}_\mathrm{G}$ is the sample variance-covariance matrix of gauge error. Based on the definitions of P/T_R and $\widehat{P/T_R}$, we have:

$$\frac{\widehat{P/T_R}}{P/T_R} = \left[\frac{|\mathbf{S}_\mathrm{G}|}{|\mathbf{\Sigma}_\mathrm{G}|}\right]^{\frac{1}{2v}}.$$

According to Anderson [1], we have:

$$\mathrm{E}\left(\frac{|\mathbf{S}_\mathrm{G}|}{|\mathbf{\Sigma}_\mathrm{G}|}\right)^{\frac{1}{2v}} = \sqrt{\frac{2}{n-1}}\frac{\prod_{i=1}^{v}\Gamma(\frac{n-p-i}{2}+\frac{1}{2v})}{\prod_{i=1}^{v}\Gamma(\frac{n-p-i}{2})}$$

Thus, the expected width of confidence interval for P/T_R can be written as:

$$P/T_R \times \sqrt{\frac{2}{n-1}}\frac{\prod_{i=1}^{v}\Gamma(\frac{n-p-i}{2}+\frac{1}{2v})}{\prod_{i=1}^{v}\Gamma(\frac{n-p-i}{2})} \times \left[\frac{1}{\left(\omega_{1-\frac{\alpha}{2}}\right)^{\frac{1}{2v}}} - \frac{1}{\left(\omega_{\frac{\alpha}{2}}\right)^{\frac{1}{2v}}}\right] \tag{5}$$

The simulation results for the optimal allocation of por parameters

The expected width of confidence interval in equation (5) can be used as a criterion for searching the optimal allocation of por parameters. The number of quality characteristics is set as $v = 3$ and the following three different scenarios are adopted in our computer simulation: (1) If the precision of measurement system is satisfactory, then we assume $P/T_R = 0.1$ (2) If the precision of measurement system is marginally acceptable, then we assume $P/T_R = 0.3$ (3) If the precision of measurement system is unacceptable, then we assume $P/T_R = 0.5$. Based on the simulation results, the optimal allocations of *por* parameters are summarized in Table 1.

TABLE 1. The optimal allocations of *por* parameters under three different scenarios for P/T_R

	(p, o, r)		
	n = 300	n = 240	n = 180
Scenario 1. ($P/T_R = 0.1$)	(10, 5, 6) (10, 6, 5) (20, 3, 5)* (20, 5, 3)* (30, 2, 5)* (30, 5, 2)*	(10, 4, 6)* (10, 6, 4)* (20, 3, 4)* (20, 4, 3)* (30, 2, 4)* (30, 4, 2)*	(10, 3, 6)* (10, 6, 3)* (20, 3, 3)* (30, 2, 3)* (30, 3, 2)*
Scenario 2. ($P/T_R = 0.3$)	(10, 5, 6) (10, 6, 5) (20, 3, 5)* (20, 5, 3)* (30, 2, 5)* (30, 5, 2)*	(10, 4, 6)* (10, 6, 4)* (20, 3, 4)* (20, 4, 3)* (30, 2, 4)* (30, 4, 2)*	X
Scenario 3. ($P/T_R = 0.5$)	(10, 5, 6) (10, 6, 5) (20, 3, 5)* (20, 5, 3)* (30, 2, 5)* (30, 5, 2)*	X	X

*denotes the alternative (n,p) combinations other than (300,10) when the inspection cost is limited.

Numerical example

To compare the appropriateness of our proposed P/T_R ratio with Majeske's P/T and Wang and Yang's P/T ratio, a numerical example adopted from Wang and Yang (2007) is used for illustration purposes. There are three quality characteristics in their solderability test of an electronic product. Ten parts (p=10) and three operators (o=3) were taken to conduct this MGRR experiment. Each operator measured all 10 parts in five consecutive trials (r=5). The measuring conditions are: the insertion speed is 20 mm/s, depth is 3 mm, and time is 5 seconds. Three values (M_1, M_2, M_3) are recorded during their measuring process, where M_1 denotes the response time (in seconds) when the device begins to solder with Sn and its lower and upper specification limits are set at (0.3, 1.0); M_2 denotes the response time (in seconds) when the solderability reaches to 2/3 maximum force and its lower and upper specification limits are set at (0.5, 1.2); M_3 denotes the maximum force (in milli-newtons, mN) during the measuring process and its lower and upper specification limits are set at (1.0, 1.2) By performing the Mardia test, we found that the p-value for Mardia SW statistic is 0.309 [5]. Thus, the assumption of multivariate normality cannot be rejected at a 95% confidence level. As the 150 randomly collected measurements follow a multivariate normal distribution, the measurement system analysis using MANOVA is then performed. We consider the random-effects MANOVA model with interaction, i.e. $\mathbf{X}_{\mathrm{ijk}} = \boldsymbol{\mu} + \mathbf{P}_i + \mathbf{O}_j + (\mathbf{PO})_{\mathrm{ij}} + \mathbf{E}_{\mathrm{ijk}}$ and calculate the correlation coefficient matrix of these three quality characteristics as shown in Table 2. Then, the three different P/T ratios including Majeske's P/T , Wang and Yang's P/T ratios, and our revised P/T_R ratio are calculated and summarized in Table 3. Note that both Majeske's P/T ratio and Wang and Yang's P/T ratio suggest the measurement systems are marginally acceptable. According to AIAG [2], the acceptance of measurement systems should be based upon importance of application measurement, cost of measurement device, costs of rework or repair, and approval by the customer. In contrast with Majeske's and Wang and Yang's MGRR results, our revised P/T ratio indicates that the measurement system is considered to be unacceptable if the product correlations among three quality characteristics are considered. From Table 2, notice that the three quality characteristics are highly correlated. Therefore, it will be more appropriate to use our revised P/T ratio to evaluate the acceptability of a multivariate measurement system.

TABLE 2. The correlation coefficient matrix of the three quality characteristics

	M_1	M_2	M_3
M_1	1	_	_
M_2	0.9999986	1	_
M_3	-0.9547479	-0.9552384	1

TABLE 3. Comparison of the three P/T ratios using different MGRR methods

	Wang and Yang (2007)	*Majeske (2008)*	*Revised P/T*
P/T	0.138	0.208	0.398

Conclusions

In this paper, we modify Majeske [4]'s P/T ratio by taking correlation coefficients among tolerances into account. The simulation results show that our revised P/T ratio outperforms the others. In the numerical example, it further indicates that both Majeske [4]'s and Wang and Yang [12]'s P/T ratios are underestimated when the quality characteristics are correlated. Hence, our proposed new $\mathrm{P}/\mathrm{T}_{\mathrm{R}}$ is the appropriate one in performing the MGRR analysis. Finally, a numerical example is given to illustrate the appropriateness of our proposed P/T ratio. A reference table with optimal allocations of parameters is also provided. Hopefully, it can be served as a useful guideline for quality practitioners when conducting multivariate GRR in industries.

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