COMPETITION OF FARMS IN DISTRIBUTING IRRIGATION WATER

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ABSTRACT

It is assumed n farms compete for the amounts of irrigation water supplied from a common source, such as a reservoir or water treatment plant. The problem is to find the optimal water amount for the farms. Applying game theoretical methodology, the noncooperative Nash equilibrium is examined. The uniqueness of the solution is proved and a simple computer method is introduced for its computation. The overall profit maximizing solution and the existence and computation of a unique solution is demonstrated. A simple example of linear profit and cost functions show the methodology when both solutions are determined. Comparison of the results illustrate the cooperative solution gives higher profits for farms than profits under competition.

INTRODUCTION

Game theory is one of the most important areas of research in mathematical economy, which has two major branches. If no cooperation is possible between the participants (called players) or if they are not willing to coordinate their actions then the game is noncooperative. The Nash equilibrium is the most appropriate solution concept in such cases where each player maximizes its own profit without any consideration to the others. In the cases of cooperative games the players not only coordinate their actions but also want to maximize their overall profit which is then divided among the players in a fair, reasonable way [3].

There are many applications of game theory in different fields of quantitative sciences such as economics and engineering. One of the most frequently studied economic models is oligopoly, which models the competition of firms producing similar products or offering similar services to a homogeneous market. A comprehensive summary of the most relevant model variants and methods are given in [1] [4] [5]. There are publications about game theory models in agriculture and water resource management [2], which are mathematically equivalent to the oligopoly models. In this paper we will introduce and solve such a model in the viewpoint of both noncooperative and cooperative game theory.

THE MATHEMATICAL MODEL

Consider n farms using irrigation water from a common source like a reservoir or a water treatment plant. The optimal water amounts given to the farms will be determined based on two different solution concepts. Let x_k denote the amount of water used by farm k, then the total water usage of all farms is given as $s = \sum_{k=1}^{n} x_k$. If $B_k(x_k)$ is the revenue of farm k from irrigation and $C(s)$ is the unit cost of water, then the profit of farm k due to irrigation is

$$\phi_k = B_k(x_k) - x_k C(s) \tag{1}$$

In this way an n-person game is defined where the n farms are the players, the strategy set of player k is an interval $S_k = [0, X_k]$, where X_k is the maximum possible water demand of farm k, and the payoff function of player k is ϕ_k. This model is equivalent to an n-firm oligopoly when the price is $-C(s)$ and the cost function of player k is $-B_k(x_k)$.

NONCOOPERATIVE SOLUTION

If the farms are not willing to coordinate their decisions, then the game is noncooperative, and the solution is the Nash equilibrium. Each farm wants to maximize its profit by maximizing

$$\phi_k = B_k(x_k) - x_k C(x_k + \sum_{l \neq k} x_l) \tag{2}$$

with respect to x_k with fixed strategies x_l of all other players ($l \neq k$). Note that

$$\frac{\partial \phi_k}{\partial x_k} = B_k'(x_k) - C\left(x_k + \sum_{l \neq k} x_l\right) - x_k C'(x_k + \sum_{l \neq k} x_l) \tag{3}$$

and

$$\frac{\partial^2 \phi_k}{\partial x_k^2} = B_k''(x_k) - 2C'\left(x_k + \sum_{l \neq k} x_l\right) - x_k C''(x_k + \sum_{l \neq k} x_l) \tag{4}$$

By assuming that

(i) $C'(s) + x_k C''(s) \geq 0$
(ii) $C'(s) - B_k''(x_k) > 0$

it is clear that the second derivative (4) is negative, so ϕ_k is strictly concave in x_k implying that the best response of farm k is unique and can be written as follows:

$$R_k(s) = \begin{cases} 0 & B_k'(0) - C(s) \leq 0 \\ X_k & B_k'(X_k) - C(s) - X_k C'(s) \geq 0 \\ x_k^* & \text{otherwise} \end{cases} \tag{5}$$

where x_k^* is the unique solution of equation

$$B_k'(x_k) - C(s) - x_k C'(s) = 0 \tag{6}$$

in interval $(0, X_k)$. If $L(x_k, s)$ denotes the left hand side, then $L_k(0, s) > 0$, $L_k(X_k, s) < 0$ and

$$\frac{\partial L_k}{\partial x_k} = B_k''(x_k) - C'(s) < 0 \tag{7}$$

by assumption (ii). Therefore there is a unique solution $x_k(s)$. Implicit differentiation of equation (6) with respect to s shows that

$$B_k''(x_k)x_k' - C'(s) - x_k'C'(s) - x_k C''(s) = 0 \tag{8}$$

implying that in the third case of (5),

$$R_k'(s) = x_k' = \frac{C'(s) + x_k C''(s)}{B_k''(x_k) - C'(s)} \leq 0 \tag{9}$$

by assuming (i) and (ii). In the first two cases of (5), $R'_k(s) = 0$, so (9) holds for all values of s. So $R_k(s)$ is a decreasing function of s for all k. Consider finally equation

$$\sum_{k=1}^{n} R_k(s) - s = 0 \tag{10}$$

where the left hand side is strictly decreasing and continuous, its value at $s = 0$ is nonnegative and at $s = \sum_{k=1}^{n} X_k$ is nonpositive, therefore there is a unique solution s^*, and the equilibrium strategies of the players are $x_k^* = R_k(s^*)$ for all k. Hence the existence of a unique Nash equilibrium is proved and a computer method is offered to find it.

COOPERATIVE SOLUTION

If the players coordinate their actions, then their logical decision is to maximize their overall profit

$$\phi = \sum_{k=1}^{n} \phi_k = \sum_{k=1}^{n} B_k(x_k) - sC(s). \tag{11}$$

Notice first that this objective function is concave as an n-variable function if all $B_k(x_k)$ are concave and $sC(s)$ is convex in s. Since

$$\left(sC(s)\right)' = C(s) + sC'(s) \tag{12}$$

and

$$\left(sC(s)\right)'' = 2C'(s) + sC''(s) > 0 \tag{13}$$

under assumption (i) and $C'(s) > 0$. So instead of (ii) we assume that

(iii) $B''_k(x_k) < 0$ and $C'(s) > 0$.

Conditions, (i) and (iii) are slightly more restrictive than (i) and (ii) assumed earlier. With given value of s, x_k is optimal when it equals

$$\bar{R}_k(s) = \begin{cases} 0 & B'_k(0) - C(s) - sC'(s) \le 0 \\ X_k & B'_k(X_k) - C(s) - sC'(s) \ge 0 \\ \bar{x}_k^* & \text{otherwise} \end{cases} \tag{14}$$

where $\bar{x}_k^*$ is the unique solution of equation

$$B'_k(x_k) - C(s) - sC'(s) = 0 \tag{15}$$

in the interval $(0, X_k)$. There is a unique solution, since the left hand side strictly decreases in x_k, its value is positive at $x_k = 0$ and negative at $x_k = X_k$. So x_k is a function of s, $x_k = x_k(s)$. Implicit differentiation of equation (15) with respect to s shows that

$$\bar{R}'_k(s) = \bar{x}'_k = \frac{2C'(s) + sC''(s)}{B''_k(x_k)} < 0 \tag{16}$$

showing that $\overline{R}_k(s)$ strictly decreases in s, so equation (10) with $\overline{R}_k(s)$ instead of $R_k(s)$ has a unique solution $\bar{s}^*$ and $\overline{R}_k(\bar{s}^*)$ is the corresponding water usage of farm k. So the existence of a unique optimal solution is proved and a computer method is offered to find it. Based on the total profit maximizing solution any cooperative solution concept such as the Core, Shapley value, Nucleolus etc. can be easily applied to find the fair share of this maximum profit.

EXAMPLE

Consider a simple linear case of $C(s) = A + Bs$ $(A, B > 0)$ and $B_k(x_k) = bx_k$ $(b > 0)$ for all players. This game is symmetric in the players, since their profit functions are identical. First we determine the noncooperative Nash equilibrium. Assuming interior optimum in (5) the third case implies so equation (6) gives the best response of the players:

$$b - (A + Bs) - x_k B = 0 \tag{17}$$

so

$$x_k = R_k(s) = \frac{b - A - Bs}{B} \tag{18}$$

and then equation (10) has the form

$$n\frac{b - A - Bs}{B} - s = 0 \tag{19}$$

so

$$s^* = \frac{n(b - A)}{B(n + 1)} \tag{20}$$

and because of the symmetry the equilibrium water usage of each farm is

$$x_k^* = \frac{b - A}{B(n + 1)} \tag{21}$$

In order to have positive equilibrium we need to assume that $b > A$.

In the case of cooperative solution equation (15) has the special form

$$b - (A + Bs) - sB = 0 \tag{22}$$

implying that

$$\bar{s}^* = \frac{b - A}{2B} \tag{23}$$

and then equal water sharing implies that the optimal water usage of each farm is

$$\bar{x}_k^* = \frac{b - A}{2Bn} \tag{24}$$

Notice that $\bar{x}_k^* < x_k^*$ for all k. The profit of each farm at the noncooperative solution is

$$\begin{aligned}\phi_k &= bx_k^* - x_k^*(A + Bs^*) = x_k^*(b - A - Bs^*) \\ &= \frac{b - A}{B(N + 1)}\left(b - A - \frac{n(b - A)}{n + 1}\right) = \frac{(b - A)^2}{B(n + 1)^2}\end{aligned} \tag{25}$$

and the profit in the case of cooperation becomes

$$\overline{\phi}_k = \overline{x}_k^*(b - A - B\,\overline{s}^*) = \frac{b-A}{2Bn}\left(b - A - \frac{b-A}{2}\right) = \frac{(b-A)^2}{4Bn}. \qquad (26)$$

It is clear that $\overline{\phi}_k > \phi_k$, since

$$4Bn < B(n+1)^2 \qquad (27)$$

which can be rewritten as

$$0 < B(n^2 - 2n + 1) = B(n-1)^2 \qquad (28)$$

CONCLUSIONS

An irrigation water distribution problem was examined, in which the optimal water amounts given to farms were determined. First the noncooperative and then the overall profit maximizing solutions were shown to be unique. These unique solutions were determined by solving single variable equations for the total water usage of all farms and then best response functions were used to find the corresponding water usage of each farm. An example with linear profit and cost functions illustrated the methodology. The actual solutions were determined and their comparison showed that it is the interest of the farms to cooperate, since their profits are higher in this case than their profits with competition.

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