A ONE-CYCLE MODEL IN SCHEDULING PREVENTIVE REPLACEMENT

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ABSTRACT

A new model is introduced to find the optimal preventive replacement policy when in addition to the preventive and failure replacement costs the generated revenue is also included in the objective function. The expected cost per unit time in a single cycle is minimized, which is the counterpart of the recently developed long-term optimization model of Finkelstein et al. [4]. The existence of finite optimum is proved and condition is given for the uniqueness of the solution. In an extension of the model including adjusted cycle length and minimal repair costs of repairable failures, conditions are given for the existence of finite optimum, which can be computed by standard computer methods.

INTRODUCTION

Finding optimal maintenance and preventive replacement policy is one of the most frequently discussed problems in reliability engineering. Nakagawa [7] gives a comprehensive survey of the most important models and solution methodologies. Most models consider the costs of minimal repairs, preventive and failure replacements in a cycle, when either the long-term expected cost per unit time or expected cost per unit time in a single cycle is minimized. A small proportion of the studies take the output of the system into consideration, and most of them assume discrete levels of the outputs. Systems with continuous output levels were studied by only a few researchers [3]. The usually degrading output level should also be included in any optimization model [2] [5] [6]. The degradation of a system has two faces, one is the increasing failure rate and the other is the decreasing output rate. The first is usually modeled by a stochastic process, like in most classical models [1], and the other by incorporating the generated revenue by the produced output into the objective function [4]. The degradation of the output level is usually considered independent of the increase of the failure rates, like the gradual "misalignment" of the system parts is independent of the hard failures of the system elements.

The long-term optimization models are based on the renewal theory, where the expected cycle cost is divided by the expected cycle length. However in the case of fast changing technology it is more appropriate to optimize for a single cycle instead of minimizing a long term average.

In a recent study Finkelstein et al. [4] developed long term cost minimizing models containing the generated revenue by the produced output, which was considered continuous, and the replacement costs. In this paper we will introduce the one-cycle counterpart of their model and the existence of finite preventive optimum will be proved. In addition computer method will be offered for finding the optimum. An extension of this model for repairable systems with adjusted cycle length is also discussed.

THE MATHEMATICAL MODEL

Consider a system, which is replaced at a scheduled time t from the last renewal point or when nonrepairable failure occurs whichever comes first. Let C_1 and C_2 be the costs of failure and preventive replacements $(C_1 > C_2)$. The CDF of time to failure is $F(t)$ with pdf $f(t)$, reliability function $R(t)$ and failure rate $\rho(t)$. The output rate $Q(t)$ is assumed to be continuous and decreasing in time. Since the time X to failure is random, the net cost per unit time in a cycle is also random:

$$g(t) = \begin{cases} \dfrac{C_2 - \int_0^t Q(x)dx}{t} & X \geq t \\ \dfrac{C_1 - \int_0^X Q(x)dx}{X} & X < t \end{cases} \tag{1}$$

where $\int_0^t Q(x)dx$ gives to accumulated output in interval $[0, t]$. The expected value of $g(t)$ is given as

$$\bar{g}(t) = \frac{\left(C_2 - \int_0^t Q(x)dx\right) R(t)}{t} + \int_0^t \frac{\left(C_1 - \int_0^x Q(\tau)d\tau\right) f(x)}{x} \, dx. \tag{2}$$

Notice that $\bar{g}(0) = \infty$ so $t = 0$ cannot be the optimum. The derivative of $\bar{g}(t)$ can be written as

$$\bar{g}'(t) = \frac{(C_1 - C_2)f(t)}{t} - \frac{(C_2 - \int_0^t Q(x)dx)R(t) + Q(t)R(t)t}{t^2} \tag{3}$$

which has the same sign as

$$h(t) = (C_1 - C_2)\rho(t) - \frac{C_2 - \int_0^t Q(x)dx + Q(t)t}{t}. \tag{4}$$

Notice that $h(0) = -\infty$ and $h(\infty) > 0$ by assuming that $\rho(0) < \infty$, $\rho(\infty) > 0$, $Q(\infty) = 0$, and the lifetime cumulated output $\int_0^\infty Q(x)dx$ is finite. Therefore there is a finite optimum. The uniqueness of the optimum however cannot be guaranteed in general, since $h(t)$ is not necessarily strictly increasing. Let $q(t)$ denote the second term of (4), then

$$q'(t) = \frac{r(t)}{t^2} \tag{5}$$

with

$$r(t) = Q'(t)t^2 - C_2 + \int_0^t Q(x)dx - Q(t)t. \tag{6}$$

Notice that $r(0) = -C_2 < 0$ and

$$r'(t) = Q''(t)t^2 + Q'(t)t. \tag{7}$$

By assuming that $Q(t)$ is concave and decreasing we have $r'(t) \leq 0$ so $r(t)$ also decreases and so $r(t) < 0$ for all t implying that $q(t)$ decreases and with constant or increasing failure rate $\rho(t)$, $h(t)$ increases. So the optimum is unique.

MODEL EXTENSIONS

Assume that the time requirements are T_1 and T_2 for failure and preventive replacements, respectively, and the cycle ends if replacements are finished and the new system starts operating. In this case the objective function is modified as

$$\bar{g}_1(t) = \frac{\left(C_2 - \int_0^t Q(x)dx\right) R(t)}{t + T_2} + \int_0^t \frac{\left(C_1 - \int_0^x Q(\tau)d\tau\right) f(x)}{x + T_1} dx. \qquad (8)$$

Assume in addition, that small repairable failures might occur during each cycle, and let $\bar{M}(t)$ denote the expected number of such failures in interval $[0, t]$ and the cost of each minimal repair is C_3. Then (8) can be further extended as

$$\bar{g}_2(t) = \frac{\left(C_2 + C_3\,\bar{M}(t) - \int_0^t Q(x)dx\right) R(t)}{t + T_2} + \int_0^t \frac{\left(C_1 + C_3\bar{M}(x) - \int_0^x Q(\tau)d\tau\right) f(x)}{x + T_1} dx. \qquad (9)$$

By selecting $C_3 = 0$, model (9) reduces to (8) and by selecting $T_1 = T_2 = 0$, model (8) becomes (2). Notice that $\bar{g}_2(0) = \frac{C_2}{T_2}$ and the sign of $\lim_{t\to\infty} \bar{g}_2(t)$ is indeterminate. The derivative of $\bar{g}_2(t)$ can be written as

$$\frac{\left[\left(C_3\bar{\rho}(t) - Q(t)\right)R(t) - \left(C_2 + C_3\bar{M}(t) - \int_0^t Q(x)dx\right) f(t)\right](t + T_2) - \left(C_2 + C_3\bar{M}(t) - \int_0^t Q(x)dx\right) R(t)}{(t + T_2)^2} + \frac{\left(C_1 + C_3\bar{M}(t) - \int_0^t Q(x)dx\right) f(t)}{t + T_1} \qquad (10)$$

where $\bar{\rho}(t)$ is the failure rate of repairable failures. It has the same sign as

$$\left[\left(C_3\bar{\rho}(t) - Q(t)\right)R(t) - \left(C_2 + C_3\bar{M}(t) - \int_0^t Q(x)dx\right) f(t)\right](t + T_2)(t + T_1)$$
$$- \left(C_2 + C_3\bar{M}(t) - \int_0^t Q(x)dx\right) R(t)(t + T_1) + \left(C_1 + C_3\bar{M}(t) - \int_0^t Q(x)dx\right) f(t)(t + T_2)^2 \qquad (11)$$

or after dividing by $R(t)$, we have

$$\bar{h}(t) = \left(C_3\bar{M}(t) - \int_0^t Q(x)dx\right)\left[\rho(t)\left((T_2 - T_1)t + T_2(T_2 - T_1)\right) - (t + T_1)\right]$$
$$+\rho(t)\left(-C_2(t + T_2)(t + T_1) + C_1(t + T_2)^2\right) + \left(\left(C_3\bar{\rho}(t) - Q(t)\right)(t + T_2)(t + T_1) - C_2(t + T_1)\right). \qquad (12)$$

If the failures follow Weibull distributions, then $\rho(0) = \bar{\rho}(0) = 0$ and $\rho(\infty) = \bar{\rho}(\infty) = \infty$, so $\bar{h}(0) = -Q(0)T_1T_2 - C_2T_1 < 0$ so $t = 0$ cannot be optimum. As $t \to \infty$ the leading terms of $\bar{h}(t)$ are as follows:

$$C_3\bar{M}(t)\rho(t)(T_2 - T_1)t + \rho(t)(C_1 - C_2)t^2 + C_3\bar{\rho}(t)t^2 \qquad (13)$$

by assuming again that the life-long output $\int_0^\infty Q(x)dx$ is finite. If η, β, and $\bar{\eta}, \bar{\beta}$ denote the parameters of the distributions of the nonrepairable and repairable failures, then the leading terms have the form

$$C_3\left(\frac{t}{\bar{\eta}}\right)^{\bar{\beta}} \quad \frac{\beta}{\eta^{\beta}}\, t^{\beta-1}(T_2 - T_1)t + \frac{\beta}{\eta^{\beta}} t^{\beta-1}(C_1 - C_2)t^2 + \frac{C_3\bar{\beta}}{\bar{\eta}^{\bar{\beta}}} t^{\bar{\beta}-1}t^2$$
$$= \frac{C_3\beta(T_2 - T_1)}{\bar{\eta}^{\bar{\beta}}\eta^{\beta}} t^{\bar{\beta}+\beta} + \frac{\beta(C_1 - C_2)}{\eta^{\beta}} t^{\beta+1} + \frac{C_3\bar{\beta}}{\bar{\eta}^{\bar{\beta}}} t^{\bar{\beta}+1}. \tag{14}$$

It is reasonable to assume that $T_1>T_2$ and $C_1>C_2$. If both β and $\bar{\beta}$ are greater than 1, then the first term dominates, so $\bar{h}(t) \to -\infty$ as $t \to \infty$. In this case there is no guarantee for finite optimum. If $t = \infty$ gives infimum, then no preventive replacement has to be scheduled, the unit is replaced upon failures. If $\beta > 1$ and $\bar{\beta} = 1$, then the two first terms are the leading terms:

$$\left[\frac{C_3\beta(T_2 - T_1)}{\bar{\eta}\eta^{\beta}} + \frac{\beta(C_1 - C_2)}{\eta^{\beta}}\right] t^{\beta+1}. \tag{15}$$

The first term in the bracketed expression is negative, the second term is positive. If their sum is negative, then $\bar{h}(t) \to -\infty$ as $t \to \infty$ giving the same conclusion as above, however if the sum is positive, then $\bar{h}(t) \to \infty$ as $t \to \infty$ implying the existence of finite, nonzero optimum. This is the case when

$$C_3(T_2 - T_1) + \bar{\eta}(C_1 - C_2) > 0. \tag{16}$$

NUMERICAL EXAMPLE

As an illustrative example let $C_1 = 200, C_2 = 100, C_3 = 10, T_1 = 0.1, T_2 = 0.05, \beta = 2,\ \eta = 5,\ \bar{\beta} = 1, \bar{\eta} = 2$, and $Q(t) = 500e^{-t}$. Here time is counted in months and costs in 100 dollars. Then $\bar{M}(t) = \left(\frac{t}{\bar{\eta}}\right)^{\bar{\beta}} = \frac{t}{2}$, $\int_0^t Q(x)dx = 500(1 - e^{-t})$, $R(t) = e^{-\left(\frac{t}{5}\right)^2}$, and $f(t) = \frac{2t}{25} e^{-\left(\frac{t}{5}\right)^2}$, so the short-term expected cost per unit time equals

$$\bar{g}_2(t) = \frac{\left(100 + 5t - 500(1 - e^{-t})\right)e^{-\frac{t^2}{25}}}{t + 0.05} + \frac{\int_0^t \left(200 + 5x - 500(1 - e^{-x})\right)2xe^{-\frac{x^2}{25}}}{25(x + 0.1)} dx \tag{17}$$

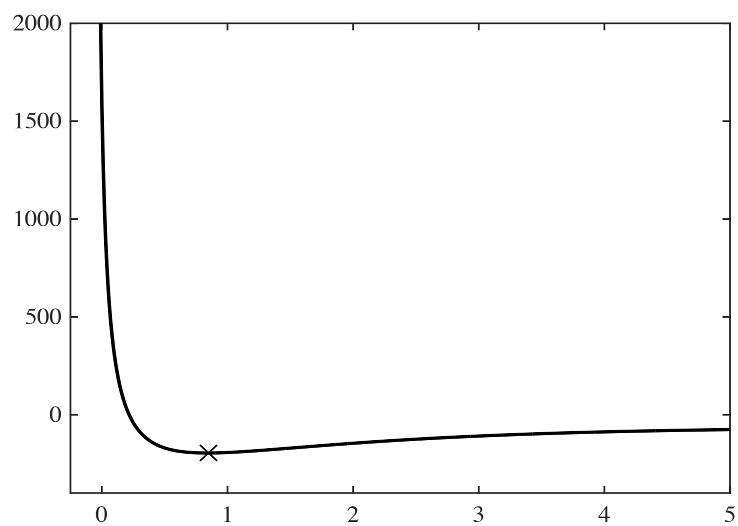

Figure 1 The shape of objective function (17)

Figure 1 shows the shape of this function, and the optimal solution becomes $t = 0.85$ month, and the optimal objective function value is -195.47, that is, in a cycle the net profit becomes 19547 dollars.

CONCLUSION

The one cycle counterpart of the model of Finkelstein [4] is first introduced to find optimal preventive replacement policy, when the cumulated output of the system is included in the objective function. The existence of the finite optimum is proved and condition is given for the uniqueness of the solution.

As an extension of this model the cycle length is adjusted and the additional costs of minimal repairs during the cycle are added to the objective function. Conditions are given for the existence of finite-optimum.

It is an interesting idea to combine long-term objectives with one-cycle objectives and solving the resulting multi-objective optimum problems. This idea will be explored in our next paper.

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