OPTIMAL SCHEDULE OF REPAIR OR REPLACEMENT AFTER DEGRADATION IS NOTICED

Maryam Hamidi, Department of Industrial Engineering, Lamar University, Beaumont, TX 77710, USA
mhamidi@lamar.edu
Akio Matsumoto, Department of Economics, Chuo University, 742-1, Higashi-Nakano, Hachioji, Tokyo, 192-0393, Japan, akiom@tamacc.chuo-u.ac.jp
Ferenc Szidarovszky, Ridgetop Group, 3580 W. Ina Road, Tucson, AZ, 85741, USA
szidarka@gmail.com

ABSTRACT

This paper introduces a new optimal repair/replacement model of production units. After failure is detected, the unit might be replaced immediately, or repaired or left on continuing to operate. In the third case the unit still operates with less output and increased operation cost. Further degradation also has an increasing effect on the repair/replacement cost. Therefore it is an important problem to determine the economically optimal length of time until repair or replacement has to be performed. A mathematical model is developed for defining the optimal solution and a practical algorithm is suggested for its computation.

Keywords: Reliability, degradation, preventive replacement, optimal scheduling.

INTRODUCTION

Age replacement problems are one of the most frequently studied problem areas in reliability engineering. Many researchers have devoted their efforts for this subject [1] [2] [3] [4] [5] [7] [8]. Models were developed under different conditions and most of them discussed the optimal replacement time of a unit or system considering the cost of repairable failures, preventive and failure replacement costs, and the status of ongoing projects. In this paper we approach this problem from a different point of view. It is assumed that a unit operates normally until a minor failure is detected, allowing the unit to continue operating with degrading output and increasing operation cost. In many practical cases it is economical to keep the unit operating instead of taking immediate action of repair or replacement. A mathematical model is developed to find the optimal length of time until such action has to be taken. In addition to the mathematical formulation of the model a solution algorithm is suggested.

THE MATHEMATICAL MODEL AND SOLUTION

An equipment is considered, which produces utility with density $u(t)$ after t time periods of operation which includes the values of its products minus all costs of operation. Assume at time T a censor shows the start of a degradation process the result of which is the decrease in utility production, which is further decreased by the increased operation costs. The repair or replacement cost also increases in time because of the increased level of degradation. In formulating the mathematical model to find the optimal time of repair or replacement the following notations are used:

α = degradation coefficient
β = cost of repair or replacement increase factor
x = planned time of repair or replacement (decision variable)

$u(t)e^{-\alpha(t-T)}$ = density of utility value produced at time $t > T$
$Ke^{\beta(x-T)}$ = cost of repair or replacement at time x
Here α shows how the utility production decreases by degradation, β represents that repair costs become larger with increased degradation level, and K is the repair cost without degradation. Accordingly, the total net utility value per unit time can be given as

$$g(x) = \frac{\int_0^T u(t)dt + \int_T^x u(t)e^{-\alpha(t-T)}dt - Ke^{\beta(x-T)}}{x} \tag{1}$$

which is maximized. Notice first that the derivative of $g(x)$ has the same sign as

$$G(x) = \left[u(x)e^{-\alpha(x-T)} - \beta Ke^{\beta(x-T)}\right]x - [\int_0^T u(t)dt + \int_T^x u(t)e^{-\alpha(t-T)}dt - Ke^{\beta(x-T)}] \tag{2}$$

Which is the numerator of the derivative of the objective function, since its denominator is positive. Clearly

$$G(T) = (u(T) - \beta K)T - \int_0^T u(t)dt + K \tag{3}$$

Let T^* be the latest time when repair or replacement has to be performed. Then

$$G(T^*) = \left[u(T^*)e^{-\alpha(T^*-T)} - \beta Ke^{\beta(T^*-T)}\right]T^* - \left[\int_0^T u(t)dt + \int_T^{T^*} u(t)e^{-\alpha(t-T)}dt - Ke^{\beta(T^*-T)}\right], \tag{4}$$

which tends to negative infinity as T^* converges to infinity.
Furthermore

$$\begin{aligned} G'(x) = {} & u'(x)e^{-\alpha(x-T)}x - \alpha u(x)e^{-\alpha(x-T)}x + u(x)e^{-\alpha(x-T)} - \beta^2 Ke^{\beta(x-T)}x - \beta Ke^{\beta(x-T)} \\ & - u(x)e^{-\alpha(x-T)} + K\beta e^{\beta(x-T)} = u'(x)e^{-\alpha(x-T)}x - \alpha u(x)e^{-\alpha(x-T)}x - \beta^2 Ke^{\beta(x-T)}x \\ & < 0 \end{aligned} \tag{5}$$

since $u(x)$ is nonincreasing. So $G(x)$ is strictly decreasing.
So we have the following cases:
a) $G(T) \leqq 0$, then $G(x) < 0$ for all $x > T$, so $g(x)$ decreases and therefore T is the optimal time.
b) $G(T^*) \geqq 0$, then $G(x) > 0$ for all $x < T^*$, so $g(x)$ increases, therefore T^* is the optimum.
c) Otherwise $G(T) > 0$ and $G(T^*) < 0$, and there is a unique $\overline{T} \in (T, T^*)$ such that $G(\overline{T}) = 0$ and this $\overline{T}$ value is the optimum.
Assume next that there are several machines with different parameters, and a common time x is to be determined to repair or replace them together in order to decrease set-up costs. In this more general case

$$g(x) = \sum \frac{\{\int_0^{T_i} u_i(t)dt + \int_{T_i}^x u_i(t)e^{-\alpha_i(t-T_i)}dt - K_i e^{\beta_i(x-T_i)}\}}{x} \tag{6}$$

where the summation is made for all machines involved. Similarly to the previous more simple case the derivative of $g(x)$ has the same sign as

$$G(x) = \left\{\sum u_i(x)e^{-\alpha_i(x-T_i)} - K_i\beta_i e^{\beta_i(x-T_i)}\right\}x - \sum\left\{\int_0^{T_i} u_i(t)dt + \int_{T_i}^{x} u_i(t)e^{-\alpha_i(t-T_i)}dt - K_i e^{\beta_i(x-T_i)}\right\} \tag{7}$$

By simple differentiation

$$\begin{aligned} G'(x) = \sum\{u_i'(x)e^{-\alpha_i(x-T_i)}x - \alpha_i u_i(x)e^{-\alpha_i(x-T_i)}x + u_i(x)e^{-\alpha_i(x-T_i)} - K_i\beta_i^2 e^{\beta_i(x-T_i)}x \\ - K_i\beta_i e^{\beta_i(x-T_i)} - u_i(x)e^{-\alpha_i(x-T_i)} + K_i\beta_i e^{\beta_i(x-T_i)}\} \\ = \sum\{u_i'(x)e^{-\alpha_i(x-T_i)}x - \alpha_i u_i(x)e^{-\alpha_i(x-T_i)}x - K_i\beta_i^2 e^{\beta_i(x-T_i)}x\} < 0 \end{aligned} \tag{8}$$

So the optimum depends on the sign of $G(T)$ and $G(T^*)$ as in the previous case.
Consider next the special case with $u_i(t) \equiv u_i$ and $\beta_i = 0$. Then

$$G(T) = uT - \beta KT - uT + K = K(1 - \beta T) \tag{9}$$

and

$$\begin{aligned} G(T^*) = [ue^{-\alpha(T^*-T)}T^* - \beta K e^{\beta(T^*-T)}]T^* - uT - u\left[\frac{e^{-\alpha(t-T)}}{-\alpha}\right]_{t=T}^{T^*} + Ke^{\beta(T^*-T)} \\ = ue^{-\alpha(T^*-T)}T^* - \beta K e^{\beta(T^*-T)}T^* - uT + \frac{u}{\alpha}e^{-\alpha(T^*-T)} - \frac{u}{\alpha} + Ke^{\beta(T^*-T)} \end{aligned} \tag{10}$$

So the formulas become much more simple.

NUMERICAL EXAMPLE

As an illustrative example let $\alpha = 0.1, \beta = 0.05, K = 7, T = 2, T^* = \infty$ and utility function be defined as $u(t) = 5$. That is, utility production decreases and repair cost increases by the exponential rates of 0.1 and 0.05, respectively. Without degradation the utility production density is assumed to be 5 and repair of the equipment is 7. It is also assumed that degradation starts at T=2 and the time horizon is infinite. It can be seen that $G(T) = 6.3 > 0$ and $G(T^*) < 0$, so the unique $\overline{T} \in (T, T^*)$ is $\overline{T} = 5.82$. The uniqueness of the solution is guaranteed by the fact that G(x) strictly decreases. The corresponding total net utility value per unit time is $g(\overline{T}) = 2.99$. Figure 1 shows the objective function with different values of repair/replacement time. The figure illustrates that after a failure is detected at $T = 2$, it is economical to keep the unit continue operating until $\overline{T} = 5.82$.

CONCLUSIONS

A new age replacement model was introduced to determine the optimal length of time until a degrading production unit is kept operating. The model was based on the degrading production density of the unit as well as the increasing operation and repair/replacement cost. The net profit per unit time in a cycle is maximized. The numerical solution of the model is based on solving a single variable monotone equation which can be performed with standard numerical methods [6].

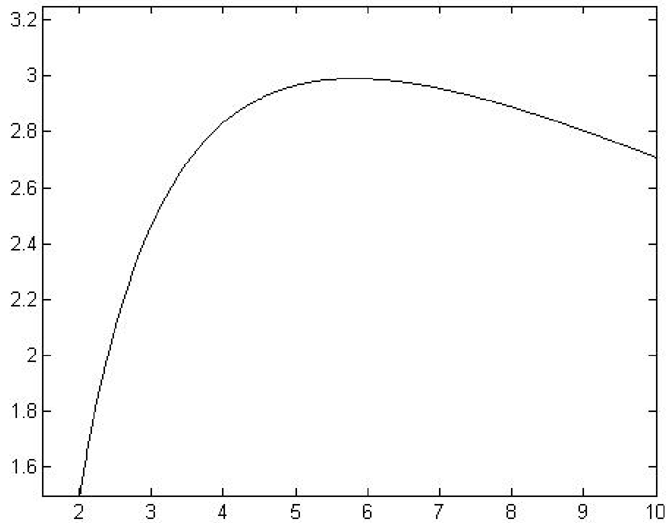

FIGURE 1. **Objective function** $\boldsymbol{g}(\boldsymbol{x})$

REFERENCES

[1] Elsayed, E. A. *Reliability Engineering*. Hoboken, John Wiley & Sons, 2012.
[2] Finkelstein, M. *Failure Rate Modelling for Reliability and Risk*. London, Springer, 2008.
[3]Jardine, A. Optimizing maintenance and replacement decisions (tutorial). Proceedings of the *Annual Reliability and Maintainability Symposium* (RAMS), Newport Beach, CA, January 23-26, 2006.
[4] Jardine, A. & Tsang, A. *Maintenance, Replacement and Reliability: Theory and Applications*. Boca Raton, Taylor and Francis, 2006.
[5] Nakagawa, T. *Advanced Reliability Models and Maintenance Policies*. London, Springer, 2008.
[6] Szidarovszky, F. & Yakowitz, S. *Principles and Procedures of Numerical Analysis*. New York, Plenum Press, 1986.
[7] Wang, H. A survey of maintenance policies of deteriorating systems. *European Journal of Operational Research*, 2002, 139, 469-489.
[8] Zhao, X. & Nakagawa, T. Optimization problems of replacement first and last in reliability theory. *European Journal of Operational Research*, 2012, 223, 141-149.