MAPPING A DECISION MAKER'S RISK ATTITUDE TO ROBUST GOAL PROGRAMMING PARAMETERS[1]

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ABSTRACT

We summarize and introduce robust goal programming (RGP) concepts that incorporate parameters corresponding to a desired level of robustness. We further discuss multiple decision maker (DM) risk elicitation methods, with an emphasis on those that utilize the constant relative risk aversion (CRRA) coefficient r as a single risk parameter. With a view towards integrating DM risk attitudes to parameterize RGP formulations, we conduct two proofs: one showing that r can be mapped into the RGP framework, and another proving that risk neutrality exists in such a framework.

INTRODUCTION

Robust goal programming (RGP), first proposed in [1] and examined more recently in the technical paper by [2], assigns values corresponding to a decision maker's (DM) risk preference in a subjective manner. Subjectively assigning a DM's risk preference induces and yields an inappropriate solution. Using proper risk elicitation methods to configure a DM's risk preference is preferable and, in doing so, this risk parameter can be mapped into the RGP framework such that r is bounded and risk neutrality is defined. A more elaborate discussion pertaining to risk elicitation methods and mapping the CRRA coefficient r into the RGP realm is conducted in the technical paper [3]. This paper introduces RGP, discusses three different risk elicitation methods, and provides two relevant proofs.

MOTIVATION

The motivation of this paper, and particularly of RGP and the risk preference mapping methodology, stems from a United States Transportation Command (USTC) rate setting problem dealing with many sources of uncertainty, as well as various goals. More specifically, USTC seeks to set transportation rates in such a way as to minimize the deviation away from zero profit, while maintaining robust rates year-to-year. Setting these rates is extremely cumbersome as they depend on many factors such as cargo classifications, distance between the origin and

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destination, amount of tonnage being shipped, and type of service being provided. Recent research has shown that USTC's current solution practices are not efficient or accurate. As a way to improve these inefficiencies, USTC seeks to disaggregate rates (i.e., account for direction and increase location codes). However, rate disaggregation presents a burden for USTC as the problem becomes more combinatorically challenging with many sources of uncertainty (supply, demand, inflation rate, years till execution, etc.), where less historical data is available to foster a probability distribution. In turn, RGP is offered as a way forward for USTC because it takes into account uncertainties by way of uncertainty sets (alleviating the need for probability distributions) and seeks to find robust solutions, while minimizing deviation from the many goals USTC has set forth. Additionally, the mapping methodology is used in conjunction with RGP ideals to take into account USTC's views on risk in terms of the USTC rate setting problem.

RGP

There are three RGP models in the literature. The first one, presented by Kuchta [1], uses cardinality-constrained robustness with interval-based uncertainty sets in conjunction with goal programming (GP) techniques. Hanks [2] also applies GP methods to construct two RGP models: (1) using cardinality-constrained robustness with norm-based uncertainty sets, and (2) using strict robustness with ellipsoidal uncertainty sets.

The Kuchta Model

The Kuchta model uses GP along with cardinality-constrained robustness and interval-based uncertainty sets. The main idea stemming from GP is to minimize total absolute deviation from some target value pertaining to multiple constraints. The GP method is first seen in [4], with influential articles in [5] [6] [7]. The notion of cardinality-constrained robustness, first presented in [8] [9] is a less conservative echelon of robustness founded on the idea of reducing the size of an interval-based uncertainty set. [8] [9] claim that it is unlikely for all coefficients within a constraint to change by their maximum-allowable deviation and suggest that only a subset of them (i.e., k_i) will be affected during any period of time. The interested reader is referred to [8] [9] for a much more in-depth discussion regarding cardinality-constrained robustness, as well as interval-based uncertainty sets. Normally, cardinality-constrained robustness allows for deviation in both the constraint matrix and objective function. However, Kuchta's model makes the assumption that only the objective function coefficients are subject to any possible deviation. Consequently, the interval-based uncertainty set used in Kuchta's model considers deviation that negatively influences the attainment of a goal exclusively. This further implies that coefficients in the constraint matrix are not subject to uncertainty (i.e., deviation). The Kuchta RGP model formulation can be seen in [1].

The Hanks Models

The Hanks models use the Kuchta model for comparison by way of strict robustness and ellipsoidal uncertainty sets, as well as cardinality-constrained robustness and norm-based uncertainty sets. Strict robustness, commonly regarded as minimax robustness, is first offered in [10]. Strict robustness is regarded as the most conservative, or risk-averse, echelon of robustness because it assumes that any deviation that can occur will do so at its maximum value. Although not ideal for all circumstances, the strict robustness formulation serves as the foundation for robust optimization (RO).

Ellipsoidal uncertainty sets have been widely used in RO, as seen in the popular work by [11] [12] [13]. Ellipsoidal uncertainty sets use a DM risk assessment parameter $\theta_i, i = 1, \dots, m$ for each goal as a way to alter the level of conservatism, which is why it is commonly used with strict robustness. [13] provides a comparison between ellipsoidal uncertainty sets with interval-based uncertainty sets by showing the largest volume ellipsoid *contained in* the box B exists when $\theta_i = 1$, whereas the smallest volume ellipsoid *containing* the box B occurs when $\theta_i = \sqrt{n}$, where n is the number of decision variables. Because of this, the Hanks model that incorporates strict robustness and ellipsoidal uncertainty sets makes the assumption that $0 \leq \theta_i \leq \sqrt{n}, \ \forall i = 1, \dots, n$.

Norm-based uncertainty sets are used in combination with cardinality-constrained robustness in the second Hanks model. An informative discussion on norm-based uncertainty sets can be seen in [14], where this particular Hanks model uses the L_1-norm and L_2-norm. The Hanks RGP models and their associated formulations can be seen in [2].

RGP methods have been utilized in practice concerning portfolio selection [15], vehicle routing [16], and supply chain management [17], to name a few.

RISK ELICITATION METHODS

Both the Kuchta and Hanks RGP models rely on a parameter corresponding to an assumption about the robustness desired, which also correlates to the risk tolerance of the DM. These models currently assign these risk tolerances subjectively, which is not ideal. Rather, the three models presented should rigorously assign a DM's risk preference by utilizing risk elicitation methods. This section presents different risk elicitation methods seen in the literature, with an emphasis on those that elicit the CRRA coefficient r. The interested reader is referred to the survey articles [18] [19] [20] for a comprehensive review of popular risk elicitation methods.

The Eckel Grossman Method

The Eckel and Grossman (EG) Method, seen in [21], is an ordered lottery that presents individuals with 5-7 lotteries, each having two choices with equal probabilities. Although the original lottery structure seen in [21] presents individuals with 5 lotteries, many include 1-2 additional lotteries to better distinguish between risk-neutral and risk-seeking behavior. The EG Method forces each individual to make exactly one lottery choice, where each lottery's expected value linearly increases as the low payoff and high payoff values decrease and increase, respectively. Upon knowing the chosen lottery, the EG Method classifies the r-value into different intervals, while using the utility function $U(x) = x^{1-r}$ to generate a risk curve, where x corresponds to wealth. Clearly, individuals with an $r > 0$ are considered risk-averse, whereas those individuals with an $r < 0$ or an $r = 0$ are classified as risk-seeking and risk-neutral, correspondingly.

The Bomb Risk Elicitation Task

The bomb risk elicitation task (BRET) is another risk elicitation method first discussed in [22]. The BRET is an interactive, choice-based game that is generally conducted on a computer. Each individual is presented with a 10x10 grid, or 100 boxes, wherein one box contains an imaginary bomb. The individuals use a guided user interface (GUI) to start and stop collecting boxes whenever they decide. For every second that passes, a box is collected and for every box collected, the individual collects some monetary value. The accumulated amount of money is displayed on the GUI, as is the number of seconds that have passed. Once the individual stops the experiment, the placement of the bomb (which is randomly selected before the experiment begins) is revealed. If the collected boxes include the bomb, the accrued money is lost. Otherwise, the individual wins the money won up to the end of the experiment. Like the EG method, the BRET assumes CRRA and uses the utility function $U(x) = x^r$, where x corresponds to wealth. Using these r-values, a DM's risk preference can be determined, where the risk-neutral individual will select 50 boxes, and a risk-averse or risk-individual will select less than or more than 50 boxes, respectively.

The Holt-Laury Method

Unlike the EG Method or the BRET, the Holt-Laury (HL) Method, discussed in [23], is considered to be a complex design risk elicitation method as it uses multiple price lists in a series of questions. The HL Method is widely used and is often considered as a baseline, or comparison, for other risk elicitation methods [18]. The HL Method asks individuals to successively choose between two lotteries, ten different times. When moving down the column for both lotteries, call them lottery A and lottery B, the payoffs remain constant, while the probabilities for both the lower and higher payoff linearly increase and decrease, respectively. Thus, at the beginning, lottery A is considered to be the safer option, where lottery B is the more risky option. However, as the probabilities change, lottery B becomes the safer option in relation to lottery A. The goal of the HL Method is to determine the crossover point; a point in which an individual changes from preferring lottery A to preferring lottery B. This crossover point is used as justification for each individual's risk preference, where the corresponding r-value intervals are determined by the "number of safe choices" (i.e., the number of times lottery A is chosen). Assuming CRRA, the HL method uses the utility function $U(x) = \frac{x^{1-r}}{(1-r)}$, where x represents wealth, to generate a risk curve. Of importance, to ensure each individual accurately choses the lottery coinciding with their risk preference, before the experiment begins each individual is told that after all lottery choices have been made, one will be randomly chosen with real monetary payoffs.

MAPPING RISK TO RGP

Clearly, the risk parameter r cannot simply be used as an RGP parameter, specifically k_i – the subjective parameter determining the number of coefficients affected by deviation. Instead, a proof is required that demonstrates r can be mapped to some value of $k_i(r)$. Since both the EG and HL Methods have $r \in (-\infty, \infty)$ and the BRET uses $r \in (0, \infty)$, we discuss the following cases for bounding r, preliminary to proving that either range of risk elicitation values can be mapped to values of a bounded function $k_i(r)$.

Case 1: $\boldsymbol{r} \in (-\infty, \infty)$

Let $S \in \mathbb{R}$, $k_i(r): S \to \mathbb{R}$, and denote S' the derived set of S (i.e., the collection of all accumulation points of S). Let $r \in (-\infty, \infty)$ where $r_{min} \in (-\infty, 0)$ and $r_{max} \in (0, \infty)$. Clearly r is not bounded as $\nexists$ some radius $q > 0$ such that $S \subseteq B(0, q)$. Consequently, consider W.L.O.G. an extremely large negative number $M^- \in S'$ and an extremely large positive number $M^+ \in S'$ such that $r_{min} \in [M^-, 0)$ and $r_{max} \in (0, M^+]$.

Case 2: $\boldsymbol{r \in (0, \infty)}$

Let $S \in \mathbb{R}$, $k_i(r): S \to \mathbb{R}$, and denote S' the derived set of S. Let $r \in (0, \infty)$ where $0 < r_{min} < r_{max} < \infty$. Clearly r is bounded below by $r = 0$, but not bounded above as $\nexists$ radius $q > 0$ such that $S \subseteq B(0, q)$. Consider W.L.O.G. an extremely large positive number P such that $M^- = (1/P) \in S'$ and an extremely large positive number $M^+ \in S'$ such that $0 < M^- \leq r_{min} < r_{max} \leq M^+$.

THEOREM 1. *Given $r \in S = [M^-, M^+]$ and a bounded function $k_i(r): S \to [0, n]$, we have $\lim_{r \to M^-} k_i(r) = 0$ exists when $\forall\, \varepsilon > 0, \exists\, \delta > 0$ such that $|k_i(r) - 0| < \varepsilon, \forall\, r \in S, r \neq M^-$ and $|r - M^-| < \delta$.*

PROOF. $\forall\, \varepsilon > 0, \exists\, \delta > 0$ such that $|k_i(r) - 0| < \varepsilon, \forall\, r \in S, r \neq M^-$ and $|r - M^-| < \delta$. Applying the triangle inequality, it is clear to see $|k_i(r) - 0| \leq |k_i(r) - (r - M^-)| + |(r - M^-) - 0| < \varepsilon$. Thus, let $\delta = \frac{\varepsilon}{|k_i(r) - (r - M^-)|}$ indicating that $\lim_{r \to M^-} k_i(r) = 0$. □

COROLLARY 1. *Given $r \in S = [M^-, M^+]$ and a bounded function $k_i(r): S \to [0, n]$, we have $\lim_{r \to M^+} k_i(r) = n$ exists when $\forall\, \varepsilon > 0, \exists\, \delta > 0$ such that $|k_i(r) - n| < \varepsilon, \forall\, r \in S, r \neq M^+$ and $|r - M^+| < \delta$.*

PROOF. Follows from the proof to Corollary 1, letting $\delta = \frac{\varepsilon}{|k_i(r) - (r - M^+ + n)|}$. □

Therefore, both the risk preference $r \in (-\infty, \infty)$ and the risk preference $r \in (0, \infty)$ can be mapped into the RGP framework where $0 \leq k_i(r) \leq n$.

RISK NEUTRALITY IN RGP

Knowing that $r \in (-\infty, \infty)$ and $r \in (0, \infty)$ can be mapped into $k_i(r)$, it appropriately follows to prove that risk neutrality exists in $k_i(r)$. Thus, consider the following:

Case 1

Define $r \in (-\infty, \infty)$ as some risk preference elicited from a continuous and monotonically increasing utility function $U(x)$. Since the risk preference $r \in (-\infty, \infty)$ can be mapped into the RGP framework where $0 \leq k_i(r) \leq n$, consider an extremely large negative number $M^- \in S'$ and an extremely large positive number $M^+ \in S'$ and assume W.L.O.G. $r \in [M^-, M^+]$. Further, denote $r_s = M^-$ as the maximum risk-seeking value and $r_a = M^+$ as the maximum risk-averse value.

Case 2

Alternatively, define $r \in (0, \infty)$ as some risk preference elicited from a continuous and monotonically increasing utility function $U(x)$. Since the risk preference $r \in (0, \infty)$ can be mapped into the RGP framework where $0 \le k_i(r) \le n$, consider an extremely large positive number P such that $M^- = (1/P) \in S'$ and an extremely large positive number $M^+ \in S'$ and assume W.L.O.G. $r \in [M^-, M^+]$. Further, denote $r_s = M^-$ as the maximum risk-seeking value and $r_a = M^+$as the maximum risk-averse value.

THEOREM 2. *Let $k_i(r)$ be a continuous function. If $r_s < r_a$ and $k_i(r_s) < h < k_i(r_a)$ then $\exists$ $r_n \in (r_s,\, r_a)$ such that $k_i(r_n) = h$.*

PROOF. Let $A = \{r \in (r_s,\, r_a)\, |k_i(r) < h\}$, let $r_n = sup\ A$, and let $\varepsilon = |k_i(r_n) - h|$. Because $k_i(r)$ is a continuous function, $\exists\ \delta > 0$ such that $|r - r_n| < \delta$ indicating that $|k_i(r) - k_i(r_n)| < \varepsilon$. By contradiction, assume that $k_i(r_n) < h$. Then $\left|k_i\left(r_n + \frac{\delta}{2}\right) - k_i(r_n)\right| < \varepsilon$, so $k_i\left(r_n + \frac{\delta}{2}\right) < k_i(r_n) + \varepsilon = h$. However, $r_n + \frac{\delta}{2} \in A$, indicating that $r_n \neq \sup A$. Further, by contradiction assume that $k_i(r_n) > h$. Then, since $r_n = sup\ A$, $\exists\ \bar{r} \in A$, where $r_n - \delta < \bar{r} < r_n$. But since $|r - r_n| < \delta$ and $|k_i(r) - k_i(r_n)| < \varepsilon$, $k_i(\bar{r}) > k_i(r_n) - \varepsilon = h$, indicating that $\bar{r} \notin A$. Therefore, $\exists\ r_n \in (r_s,\, r_a)$ such that $k_i(r_n) = h$, where r_n is some risk-neutral value. □

CONCLUSION

RGP is a relatively new concept, explored to date via only three models in the literature. Herein, we propose the parameterization of these models to a DM utilizing a risk preference parameter, as elicited by a rigorous and well-established method from the literature, and we establish the theoretical foundation to map risk elicitation parameters for use in RGP. Given some risk parameter value r, the mathematics to formally assign a $k_i(r)$-value has yet to be determined, but knowing that r-values map to bounded values of $k_i(r)$ and that there exists some risk neutral value between risk-averse and risk-seeking endpoints will assist in the process.

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