# OPTIMAL IMPERFECT MAINTENANCE OF A MACHINE SHOP WITH DIFFERENT VIRTUAL AGES

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## ABSTRACT

A machine shop is considered with machines having different virtual ages and failure distributions. Performing maintenance on any one of the machines decreases its virtual age by a given factor resulting in lower probability of the occurrence of later non-repairable failures. Due to limited maintenance budget the optimal choice of machines to be maintained has to be determined, which minimizes the total expected maintenance and replacement costs.

**Keywords:** Virtual age, non-repairable failures, imperfect maintenance

## INTRODUCTION

In any machine shop the management has to consider the occurrence of the random machine failures. In order to reduce the frequency of failures, regular maintenance program is expected.

In reliability engineering the random failures are characterized by the probability distributions of the time to failure. When a machine is idle (for example because of lack of jobs), then it is not degrading, so its virtual age has to be considered, which is the total time length when the machine was actually working.

In order to decrease the frequencies of failures, the management has to establish a regular maintenance program. In the literature three types of maintenance or repair actions are discussed. Minimal maintenance does not change the virtual age of the machine, it only eliminates the cause of failure. The other extreme is preventive replacement, when the machine is replaced by a new machine with zero virtual age. Partial repair is between these extreme cases when the virtual age becomes shorter, so after repair the machine will behave like a younger machine. A comprehensive summary of the main concepts of reliability theory, maintenance, different types of repair and preventive replacement policies are discussed in [1] [2] [3] [4] [5] and [6] among others.

In this paper a simple mathematical model is introduced, which can help the management of a machine shop to find the optimal choice of the machines to be maintained in order to minimize the expectation of the total maintenance and possible failure replacement costs. The replacement costs include material and labor costs as well as possible damages resulting by the failures.

## THE MATHEMATICAL MODEL

A machine shop is considered with N machines. They are subject to random non-repairable failures, when failure replacements have to be performed with costs $C_{kf}$ $(k = 1, 2 \ldots N)$. From historical failure data the CDF of time to failure is known for each machine, which is denoted by $F_k(t)$.
At a certain time, when maintenance decisions are made, the virtual age of each machine, $T_k$ $(k = 1, 2 \ldots N)$, is also known, which is the total time when the machine was actually working. The maintenance decreases the virtual age of machine k by a constant factor $\alpha_k$ $(0 \leq \alpha_k \leq 1)$, when $\alpha_k = 0$ means preventive replacement and $\alpha_k = 1$ corresponds to minimal maintenance. So if maintenance is done on machine k, then its virtual age, $T_k$, changes to $\alpha_k T_k$. The maintenance cost of machine k is assumed to be $c_k$. The management wants to decide on the maintenance strategy by selecting the machines to be maintained so that the sum of the maintenance costs and the expected total replacement costs is minimal during the next T time periods.
Consider now machine k with virtual age $T_k$. If no maintenance is preformed then the conditional CDF of time to failure equals

$$F_k^c(t) = \frac{F_k(t) - F_k(T_k)}{1 - F_k(T_k)} \tag{1}$$

So the probability that failure occurs during the next T time periods is given as

$$p_k^c = \int_{T_k}^{T_k+T} \frac{f_k(t)}{1 - F_k(T_k)} dt = \frac{F_k(T_k+T) - F_k(T_k)}{1 - F_k(T_k)} \tag{2}$$

where $f_k(t) = \frac{d}{dt} F_k(t)$. However if maintenance is done, then similarly to the previous case the conditional CDF of time to failure becomes

$$\bar{F}_k^c(t) = \frac{F_k(t) - F_k(\alpha_k T_k)}{1 - F_k(\alpha_k T_k)} \tag{3}$$

and the probability of failure during the next T time periods is similarly

$$\bar{p}_k^c = \int_{\alpha_k T_k}^{\alpha_k T_k + T} \frac{f_k(t)}{1 - F_k(\alpha_k T_k)} dt = \frac{F_k(\alpha_k T_k + T) - F_k(\alpha_k T_k)}{1 - F_k(\alpha_k T_k)} \tag{4}$$

The expected replacement cost for machine k is either $p_k^c C_{kf}$ (if no maintenance is done) or $\bar{p}_k^c C_{kf}$ (if maintenance is preformed).
The decision maker has the following decision variables:

$$x_k = \begin{cases} 1 & \text{if maintenance is done on machine k} \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

So the mathematical model is formulated as a {0, 1} binary programming problem:

$$\text{Minimize} \sum_{k=1}^{N} \{x_k(c_k + \bar{p}_k^c C_{kf}) + (1 - x_k) p_k^c C_{kf}\} \tag{6}$$

Assume in addition that the maintenance budget is $B^m$, so we have to require that

$$\sum_{k=1}^{N} x_k c_k \le B^m \tag{7}$$

This optimization problem can be solved by routine methods. Notice that the objective function can be written as:

$$\sum_{k=1}^{N} p_k^c C_{kf} + \sum_{k=1}^{N} x_k ( c_k + ( \bar{p}_k^c - p_{k)}^c C_{kf} \} \tag{8}$$

where the first term is a constant, does not depend on the decision variables.
In reliability engineering Weibull distributions are mostly used to model random times to failure, when

$$F_k( t) = 1 - e^{-\left(\frac{t}{\eta_k}\right)^{\beta_k}} \qquad (k = 1, 2, \dots, N) \tag{9}$$

where $\eta_k$ and $\beta_k$ are the model parameters, which can be obtained from earlier failure observations by using the maximum likelihood method. Then the corresponding pdf becomes:

$$f_k( t) = \frac{\beta_k t^{\beta_k - 1}}{\eta_k^{\beta_k}} e^{-(\frac{t}{\eta_k})^{\beta_k}} \tag{10}$$

Therefore from relation (2):

$$p_k^c = \frac{e^{-(\frac{T_k}{\eta_k})^{\beta_k}} - e^{-(\frac{T_k+T}{\eta_k})^{\beta_k}}}{e^{-(\frac{T_k}{\eta_k})^{\beta_k}}} \tag{11}$$

And

$$\bar{p}_k^c = \frac{e^{-(\frac{\alpha_k T_k}{\eta_k})^{\beta_k}} - e^{-(\frac{\alpha_k T_k+T}{\eta_k})^{\beta_k}}}{e^{-(\frac{\alpha_k T_k}{\eta_k})^{\beta_k}}} \tag{12}$$

In other types of failure distributions the CDF and therefore the above integrals cannot be obtained analytically, numerical integration has to be used. A comprehensive summary of the most popular methods is given for example in [7]. The solution methodology for linear binary optimization problems is discussed for example in [8] and [9].

## NUMERICAL EXAMPLE

Consider five machines with Weibull failure distributions with parameters $\eta_k = 5$ and $\beta_k = 3$ for $1 \le k \le 5$ . Assume that the virtual ages of the machines are $T_1 = 2, T_2 = T_3 = 3$ and $T_4 = T_5 = 4$. The maintenance costs are assumed to be $c_1 = c_2 = c_3 = 4$ and $c_4 = c_5 = 5$ and the virtual age decreasing factors are $\alpha_1 = \alpha_3 = \alpha_5 = 0.4$ and $\alpha_2 = \alpha_4 = 0.2$. The selected time period has the length of $T = 4$.
With these parameter selections the values of $p_k^c$ and $\bar{p}_k^c$ are computed as shown in Table 1.

| k | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $p_k^c$ | 0.81062 | 0.92018 | 0.92018 | 0.97224 | 0.97224 |
| $\bar{p}_k^c$ | 0.5855 | 0.5402 | 0.6708 | 0.5855 | 0.7464 |

**Table 1. Failure probabilities**

The maintenance budget is given as $B^m = 15$. The failure replacement costs are $C_{1f} = C_{2f} = 15$ and $C_{3f} = C_{4f} = C_{5f} = 20$ . In minimizing objective (8) subject to constraint (7) GAMS software is used, the optimal solutions are given in Table 2.

| k | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $x_k$ | 0 | 1 | 1 | 1 | 0 |

**Table 2. Optimal solutions**

The results show that machines 2, 3 and 4 have to be maintained.

## CONCLUSIONS

Optimal maintenance programs can be determined by solving the linear binary optimization problem suggested in the paper. Maintaining any one of the machines has a given cost and decreases its virtual age resulting in lower probability of a non-repairable failure in a considered future time interval. The maintenance costs and expected savings in future possible replacements require the determination of an optimal tradeoff. The mathematical model requires the probability distributions of time to failures, maintenance costs and virtual age decreasing factors, the current virtual ages of the machines, replacement costs including failure damages, and maintenance budget. The expected total maintenance and replacement cost of the entire machine shop is minimized giving the optimal maintenance decision to the management.

The basic model introduced in this paper can be easily extended in several ways. The repair costs of repairable failures can be also included in the objective function, when the expected number of such failures can be obtained based the failure rate depending on the virtual age of the machines at the beginning of the considered time interval. Instead of expected cost we could also consider expected cost per unit time, when the cycle ends either at time T or at the time of non-repairable failure, which occurs first. These model variants will be the subject of our next project.

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