AN IMPROVED EOQ MODEL FOR INVENTORY MANAGEMENT

ABSTRACT

The Economic Order Quantity (EOQ) model is one of the oldest and well known models in operations research. The standard model assumes that the inventory holding cost is proportional to the monetary value of the inventory and expressed on a percentage per year basis. It is calculated as a simple interest on continuously varying inventory levels over time. However, this ignores the compounding effects that usually happens when funds are invested, i.e. interest on interest. In this research, we extend the EOQ model to account for compounding and show that the standard model underestimates the total annual costs.

Keywords: economic order quantity, EOQ, inventory, replenishment, order quantity

INTRODUCTION

The Economic Order Quantity (EOQ) model is the most basic of all inventory models and has been widely used in practice in its standard as well as extended versions (Nahmias and Olsen [11]). Even though the model is so basic and appears in all elementary as well as advanced texts in operations research and operations management, most people do not know the origins of it. The full history of the model has been detailed in Erlenkotter [4]; according to his research on the origins of the model, the basic EOQ model was developed by Ford Whitman Harris in his seminal paper (Harris [6]). It is based on the tradeoff between the inventory holding cost that typically represents the opportunity cost of capital invested in the inventory and the ordering cost that is incurred with each inventory order. If the order quantity (Q) is small, then there are more frequent orders which results in large annual ordering cost, on the other hand, if Q is large, the average inventory over time is large and this results in a large annual inventory holding cost. The EOQ model finds the optimal order quantity that minimizes the sum of annual ordering and inventory holding costs. The standard model is based on a simple interest calculated on the average inventory level over time, to represent the opportunity cost of capital invested in the inventory. This ignores the effect of compounding, i.e. the fact that interest is earned on interest as well as the initial capital, because the interest is added to the initial investment at the time of interest calculation. In this paper, we present a model that is based on compound interest and finds the optimal order quantity based on a more accurate estimation of annual total cost.

LITERATURE REVIEW

The standard EOQ model has been extended by numerous researchers over the years to account for realistic situations such as multiple products and joint ordering, finite storage capacity, stochastic demand and stochastic lead times and the cost of backordering, among others. For instance, Donaldson [2], Silver [19] and Ritchie [15] extend the EOQ model to the case where the demand is increasing linearly over time. Reddy and Ranganatham [14] extend the model to exponentially increasing demand. Buzacott [1] relaxes the constant price assumption and studies the impact of inflation on EOQ. Erel [3] also studies the impact of continuous price increases due to inflation. The EOQ model has also been extended to quantity discounts and the standard approach appears in all elementary texts in operations management. Hwang [7] extends the model to quantity discounts on both purchasing cost and freight cost. Joint replenishment of multiple items is another area of research for EOQ extensions. Since the early work of Shu [17] and Nocturne [12] who studied two items, joint replenishment problems were intensively studied. Silver [18] and Goyal [5] were among the first to consider more than two items and propose heuristic solution algorithms. Khouja et al. [9] study the joint replenishment problem with continuously changing price. Salameh et al. [16] extend the joint replenishment problem to a situation in which an item can be substituted by another if it is stocked out. Khouja and Goyal [8] provides a review of the literature on the joint replenishment problem. There are many other extensions to the EOQ model, but to our knowledge, no other researcher extended the standard simple interest computation of the inventory holding cost to compound interest. Porteus [13] considers the time value of money and provides an approximation to the Discounted Average Value (DAV) in regenerative models in general and applies it to the EOQ model. However, his proposed model doesn't explicitly include a compounded inventory holding cost, but he discounts the cash flows at the beginning of each ordering cycle to present time and then calculates an approximate DAV based on that NPV. Mahajan and Diatha [10] extends this model to perishable products. Stuart [20] analyzes the model of Porteus [13] and provides some more insights into it for graduate teaching purposes. In this paper, we take a different approach. We do not use discounted average value of cash flows, but directly extend the EOQ model by replacing the simple interest in the model with continuously compounded interest, which is a more closely related model to EOQ than the former.

PROBLEM DESCRIPTION

In the basic EOQ model, the demand (D) is constant and deterministic and occurs at a constant rate over time. There is an ordering cost (S) that is incurred with every order, the order quantity (Q) is the same for every order and the inventory holding cost is expressed as a constant fraction per dollar per year (r). The unit purchasing cost of the item is c. Then, the total cost per year can be expressed as a function of order quantity Q as follows:

$$TC(Q) = \frac{D}{Q}S + rc\frac{Q}{2}. \tag{1}$$

Eq.1 is convex, so setting the first derivative with respect to Q equal to zero gives the famous square root equation for the optimal order quantity that minimizes the annual total cost, which is called the Economic Order Quantity (EOQ):

$$EOQ = \sqrt{\frac{2DS}{rc}}. \tag{2}$$

The second term of Eq. 1 is the annual average inventory holding cost, and it is calculated by applying an annual simple interest rate of r to the varying inventory levels over an inventory cycle and multiplying it by the number of cycles per year (D/Q). The inventory level over an inventory cycle varies according to $I(t) = Q - Dt$, for time t starting from the beginning of the cycle and $0 \leq t \leq D/Q$, where D/Q is the length of a cycle. Thus, the annual inventory holding cost is calculated as follows:

$$\frac{D}{Q}\int_0^{Q/D} (Q - Dt)\ rc\ dt = rc\frac{Q}{2}. \tag{3}$$

But in practice, an amount of money invested in a bank or investment account typically earns compound interest, depending on the compounding period. So the above equation is an underestimation of the true inventory holding cost. With continuous compounding, the annual inventory holding cost can be calculated as follows:

$$H = \frac{D}{Q}\int_0^{Q/D} (Q - Dt)\; rc\; e^{r\left(\frac{Q}{D}-t\right)}\; dt = Drc\int_0^{Q/D} e^{r\left(\frac{Q}{D}-t\right)}\; dt - \frac{D^2 rc}{Q}\int_0^{Q/D} t\, e^{r\left(\frac{Q}{D}-t\right)}\; dt$$

$$H = D\,c\;e^{r\frac{Q}{D}} + \frac{D^2\,c}{r\,Q} - \frac{D^2\,c}{r\,Q}e^{r\frac{Q}{D}}. \tag{4}$$

In order to calculate the annual total cost, we need to add the annual ordering cost to Eq. 4, which results in the following equation:

$$TC(Q) = \frac{D}{Q}S + D\,c\,Q\;e^{r\frac{Q}{D}} + \frac{D^2\,c}{r\,Q} - \frac{D^2\,c}{r\,Q}e^{r\frac{Q}{D}}. \tag{5}$$

Theorem 1. *Eq. 5 is convex with respect to Q in the interval $Q \geq 0$.*

Proof:

$$\frac{d\,TC(Q)}{d\,Q} = -\frac{D\,S}{Q^2} + rc\,e^{r\frac{Q}{D}} - \frac{D^2\,c}{r\,Q^2} + \frac{D^2\,c}{r\,Q^2}\,e^{r\frac{Q}{D}} - \frac{Dc}{Q}\,e^{r\frac{Q}{D}}. \tag{6}$$

After some algebraic manipulations and a change of variables for $x = r\frac{Q}{D}$, this results in:

$$f(x) = e^x(x-1)^2 + rx - \frac{rS}{D} - 1. \tag{7}$$

Then, we can calculate the second derivative of TC as follows:

$$\frac{d^2\,TC(Q)}{d\,Q^2} = \frac{d\,f(x)}{d\,x}\frac{d\,x}{d\,Q} = (e^x(x-1)^2 + 2xe^x + r)\frac{r}{D} > 0, \quad \forall\; Q \geq 0. \quad \square \tag{8}$$

In order to find the optimal Q, we need to set Eq. 7 equal to zero and solve for Q. This can be accomplished very easily using Newton's method and then the optimal order quantity Q could be determined by the substitution $Q^* = \frac{D}{r}x$.

EXPERIMENTAL RESULTS

It is fairly obvious that the annual total cost for the EOQ model with compound interest would be greater than the annual total cost according to the standard EOQ model. However, what is not known is how much the difference would be and how compounding affects the optimal order quantity. We calculated the optimal order quantity for varying values of D, S, r and c using Newton's method and compared it to the standard EOQ. We also calculated total costs with each method and the results are shown in Tables 1-2, where EOQ refers to the standard EOQ, EOQ_C refers to the EOQ with compounding, TC refers to the total cost that is calculated with compounding but for the standard EOQ, and TC_C refers to the total cost with compounding for the EOQ with compounding. As can be seen from the results, the total costs do not change significantly for EOQ vs. EOQ_C. However, there is some significant change in the optimal order quantity. The optimal order quantity from the compounding model is lower than the standard EOQ in all cases, as much as 5%. It is interesting to note that as the order quantity gets larger, far above

D	S	c	r	EOQ	EOQ_C	TC	TC_C
500	100	10	0.1	316.23	303.75	323.06	322.78
500	100	10	0.2	223.61	211.45	461.01	460.22
500	100	10	0.3	182.57	170.65	568.57	567.14
500	100	10	0.4	158.11	146.37	660.43	658.23
500	100	10	0.5	141.42	129.84	742.28	739.20
1000	100	10	0.1	447.21	434.50	453.99	453.79
1000	100	10	0.2	316.23	303.75	646.11	645.55
1000	100	10	0.3	258.20	245.89	795.19	794.17
1000	100	10	0.4	223.61	211.45	922.01	920.44
1000	100	10	0.5	200.00	187.96	1034.62	1032.43
10,000	100	10	0.1	1414.21	1401.08	1420.92	1420.85
10,000	100	10	0.2	1000.00	986.95	2013.43	2013.26
10,000	100	10	0.3	816.50	803.51	2469.68	2469.35
10,000	100	10	0.4	707.11	694.17	2855.38	2854.88
10,000	100	10	0.5	632.46	619.57	3196.01	3195.31

Table 1: The optimal order quantity and total cost for different demand levels

D	S	c	r	EOQ	EOQ_C	TC	TC_C
500	50	10	0.1	223.61	217.25	226.99	226.89
500	50	10	0.2	158.11	151.88	323.06	322.78
500	50	10	0.3	129.10	122.95	397.60	397.09
500	50	10	0.4	158.11	151.88	323.06	322.78
500	50	10	0.5	129.10	122.95	397.60	397.09
500	20	10	0.1	141.42	138.83	142.77	142.74
500	20	10	0.2	100.00	97.45	202.71	202.64
500	20	10	0.3	81.65	79.12	249.02	248.89
500	20	10	0.4	70.71	68.20	288.29	288.09
500	20	10	0.5	63.25	60.75	323.06	322.78
500	10	10	0.1	100.00	98.70	100.67	100.66
500	10	10	0.2	70.71	69.42	142.77	142.74
500	10	10	0.3	57.74	56.45	175.23	175.25
500	10	10	0.4	50.00	48.72	202.71	202.63
500	10	10	0.5	44.72	43.45	227.00	226.90

Table 2: The optimal order quantity and total cost for different ordering costs

the optimal quantity, the difference in total costs for the compounding model and the standard model gets larger, as shown in Figure 1. The standard EOQ model significantly underestimates total costs when Q is large.

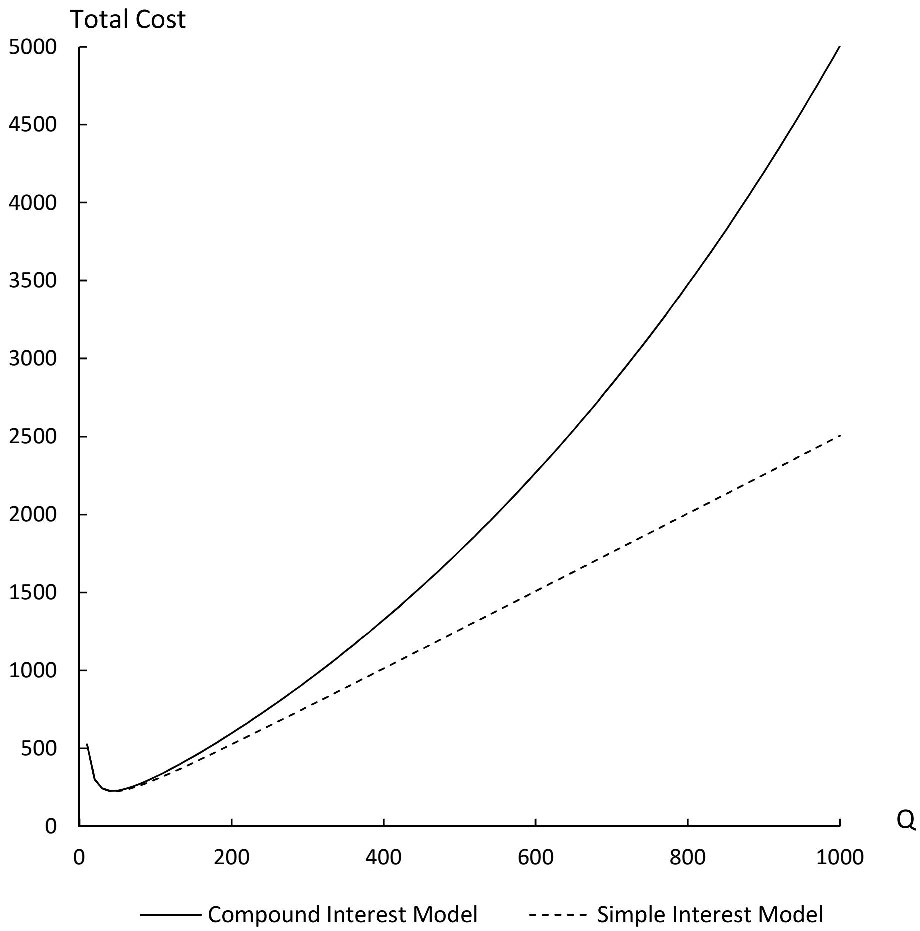

Figure 1: Total cost curves of the Simple Interest Model (Standard EOQ) vs. the Compound Interest Model where $D = 500$, $S = 10$, $c = 10$ and $r = 0.5$

CONCLUSIONS

In this research, we extend the standard EOQ model of the inventory management to the case where inventory holding cost is based on compound interest as opposed to simple interest. Even though we could not find a closed form equation for the optimal order quantity, it is fairly simple to calculate it using a simple spreadsheet or a simple program using Newton's method. The optimal order quantity is lower with the compounding method compared to the standard EOQ model. However, the difference in annual total cost is not very significant, though it is smaller with the compounding model as opposed to EOQ. The standard EOQ model significantly underestimates total costs when the order quantity is large. Further research is needed to see the full effect of compounding by varying the parameters in different ranges.

REFERENCES

[1] J.A. Buzacott. Economic order quantities with inflation. *Operational Research Quarterly*, 26 (3):553–558, 1975.

[2] W. A. Donaldson. Inventory replenishment policy for a linear trend in demand – an analytical solution. *Journal of the Operational Research Society*, 28(3):663–670, 1977.

[3] E. Erel. The effect of continuous price change in the EOQ. *Omega*, 20(4):523–527, 1992.

[4] D. Erlenkotter. Ford Whitman Harris and the economic order quantity model. *Operations Research*, 38(6):937–946, November-December 1990.

[5] S. K. Goyal. Determination of optimum packaging frequency of items jointly replenished. *Management Science*, 21(4):436–443, 1974.

[6] F.W. Harris. How many parts to make at once. *Factory, The Magazine of Management*, 10 (2):135–136, 152, February 1913.

[7] H. Hwang. An EOQ model with quantity discounts for both purchasing price and freight cost. *Computers & Operations Research*, 17(1):73 – 78, 1990.

[8] M. Khouja and S. Goyal. A review of the joint replenishment problem literature: 1989-2005. *European Journal of Operational Research*, 186(1):1–16, 2008.

[9] M. Khouja, S. Park, and C. Saydam. Joint replenishment problem under continuous unit cost change. *International Journal of Production Research*, 43(2):311–326, 2005.

[10] S. Mahajan and K.S. Diatha. Minimizing the discounted average cost under continuous compounding in the EOQ models with a regular product and a perishable product. *American Journal of Operations Management and Information Systems*, 3(2):52–60, June 2018.

[11] S. Nahmias and T.L. Olsen. *Production and Operations Analysis*. Waveland Press, 7th edition, 2015.

[12] D.J. Nocturne. Economic ordering frequency for several items jointly replenished. *Management Science*, 19(9):1093–1096, 1973.

[13] E.L. Porteus. Undiscounted approximations of discounted regenerative models. *Operations Research Letters*, 3(6):293 – 300, 1985.

[14] B.C. Reddy and G. Ranganatham. An EOQ model with exponentially increasing demand under two levels of storage. *Vision*, 16(2):121–127, 2012.

[15] E. Ritchie. The E.O.Q. for linear increasing demand: A simple optimal solution. *The Journal of the Operational Research Society*, 35(10):949–952, 1984.

[16] M.K. Salameh, A.A. Yassine, B. Maddah, and L. Ghaddar. Joint replenishment model with substitution. *Applied Mathematical Modelling*, 38(14):3662–3671, 2014.

[17] F.T. Shu. Economic ordering frequency for two items jointly replenished. *Management Science*, 17(6):B406–B410, 1971.

[18] E.A. Silver. Simple method of determining order quantities in joint replenishments under deterministic demand. *Management Science*, 22(12):1351–1361, 1976.

[19] E.A. Silver. A simple inventory replenishment decision rule for a linear trend in demand. *The Journal of the Operational Research Society*, 30(1):71–75, 1979.

[20] H.W. Stuart. Financing terms in the EOQ model. Teaching Note, Columbia Business School, August 2004.