ALGEBRAIC EXPRESSIONS FOR RANGE CONTROL CHART CONSTANTS

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ABSTRACT

A principal goal of Six Sigma is process improvement. Control charts designed to monitor the variance of a process, including the range chart, are especially well-suited to find opportunities to reduce process variation. Typically, range-chart design requires the use of tabulated constants, which have been computed using numerical integration. We provide exact results for range-control-chart constants for sample sizes of 2 and 3 as well as extremely accurate rational function approximations for sample sizes from 4 to 1000. Our expressions remove the need for table lookup in the control chart design phase, thereby simplifying the procedure in a computer-based system.

Keywords: Quality, Six Sigma, SPC, Rational Functions

INTRODUCTION

One of the basic tenets of Six Sigma is continuous process improvement (CPI) through the reduction of process variation. In fact, Dr. Edward Popovich has defined Six Sigma as, "The relentless and rigorous pursuit of the reduction of variation in all critical processes to achieve continuous and breakthrough improvements that impact the bottom line and/or top line of the organization and increase customer satisfaction." [3] By definition, removal of variation makes process output more precise, which in turn reduces waste and saves money.

Statistical process control (SPC) methods help in the diagnosis of process problems. A common SPC tool is the process control chart, and probably the most common control chart used to monitor process variation is the range (or R) control chart. The R chart is widely used in practice because of its computational simplicity and its ease of interpretation. In one respect, however, the design of the chart is not quite so simple. In particular, the computation of control chart limits requires the use of control chart constants that are typically looked up in a table. These constants were derived from two fundamental constants, d_2 (the expected value of the range of standard normal random variables) and d_3 (the standard deviation of the range of standard normal random variables). The tabulated values of d_2 and d_3 have been found by numerical integration because analytical solutions have not been found. While the design procedure is not difficult, it is somewhat of a black box, and it requires table lookup procedures to implement on a computer.

In this paper, we provide simple expressions that obviate the need for table lookups. For sample sizes of 2 and 3, the provided expressions are exact. For larger sample sizes (up to 1000) the expressions are rational function (of the sample size) approximations. These expressions are simple to implement on a computer, and very accurate. In the next section, we review the general expressions for the distribution, mean and standard deviation of the range, and then solve the general expressions for sample sizes of 2 and 3. In the following section, we discuss the approximation procedure, and present several possible rational functions that can be used to approximate d_2 and d_3 for sample sizes larger than 3. In the last section, we conclude.

THE DISTRIBUTION OF THE RANGE

The control chart constants used to design the range control chart are all derived from the distribution of the range of normal random variables. The usual assumption in SPC is that the process generates random output that follows a normal distribution. It is not our purpose here to explore violations of this assumption (see [1], [2] and [9] for example discussions of potential problems when normality is not observed). All of the control chart constants associated with the range chart can be found if the mean and variance of the range are known. The mean of the range of normal random variables is denoted by $d_2\sigma_x$, where σ_x is the standard deviation of the process; the standard deviation of the range is represented by $d_3\sigma_x$. In the remainder of the paper we will assume without loss of generality that the normal random output variable has mean 0 and standard deviation 1. Hence the design of the range control chart (and associated $\overline{X}$ chart) is equivalent to finding the constants d_2 and d_3.

To find the mean and variance of the range, we can begin with an expression for the probability density function (pdf) of R, $f_R(r)$. The pdf of the range can be found by integrating the midrange T out of the joint distribution of R and T [5, p. 255].

$$f_R(r) = n(n-1)\int_{-\infty}^{\infty}\left[\Phi\left(t+{r}/{2}\right)-\Phi\left(t-{r}/{2}\right)\right]^{n-2}\varphi\left(t+{r}/{2}\right)\varphi\left(t-{r}/{2}\right)dt\,, \tag{1}$$

where n is the sample size, and Φ and φ are the distribution function and pdf of a standard normal random variable. From (1) we can find the control chart constants d_2 and d_3 as

$$d_2=\int_0^{\infty} r f_R(r)dr \tag{2}$$

and

$$d_3=\sqrt{\int_0^{\infty} r^2 f_R(r)dr - \mathrm{d}_2^2}\,. \tag{3}$$

For general n, the expressions (1) - (3) are difficult to solve analytically. Hence, the control chart constants d_2 and d_3 that are tabulated in many quality control handbooks and textbooks were found using numerical integration. It is possible, however, to find analytical solutions for sample sizes 2 and 3.

Solution for samples of size 2

When n=2, (1) simplifies substantially. We recognize that the solution to this case is well known, but we present its solution briefly for completeness. More complete details of the derivation are available from the author. First note that

$$\varphi\left(t+{r}/{2}\right)\varphi\left(t-{r}/{2}\right)=\frac{1}{2\pi}\exp(-t^2)\exp\left(-\frac{r^2}{4}\right) \tag{4}$$

from the definition of the standard normal pdf. For n=2, (1) therefore simplifies so that

$$f_R(r)=\frac{1}{\pi}\exp\left(\frac{-r^2}{4}\right)\int_{-\infty}^{\infty}\exp(-t^2)dt \tag{5}$$

The integral portion of (5) is $\sqrt{\pi}$, so the pdf of the range when n=2 is

$$f_R(r) = \frac{1}{\sqrt{\pi}} \exp\left(\frac{-r^2}{4}\right). \tag{6}$$

Substituting (6) into (2) and (3) allows d_2 to be found by direct integration and d_3 by integration by parts. The results are that $d_2 = 2/\sqrt{\pi}$ and $d_3 = \sqrt{2 - \frac{4}{\pi}}$.

Solution for samples of size 3

When n=3, the analytical solutions to (1) - (3) are not as straightforward as with n=2, and to our knowledge have not been reported in the literature[1]. We again provide a brief derivation here; more details are available from the author upon request. To find the solution for n=3 we can rewrite (1) as

$$f_R(r) = \frac{n(n-1)}{2\pi} \exp\left(\tfrac{-r^2}{4}\right) \int_0^r \frac{dI(r)}{dr} dr . \tag{7}$$

where

$$\frac{dI(r)}{dr} = \int_{-\infty}^{\infty} (n-2) \exp\left(-t^2\right) \left[\Phi\left(t + \frac{r}{2}\right) - \Phi\left(t - \frac{r}{2}\right)\right]^{n-3} \left\{\frac{1}{2}\left[\varphi\left(t + \frac{r}{2}\right) + \varphi\left(t - \frac{r}{2}\right)\right]\right\} dt . \tag{8}$$

By substituting n=3, expanding the φ functions, combining the exponential terms, and completing the square we can create two normal pdfs as integrands in (8), which allows the terms to be integrated out to obtain

$$\frac{dI(r)}{dr} = \frac{1}{\sqrt{3}} \exp\left(\tfrac{-r^2}{12}\right) . \tag{9}$$

Plugging (11) into (7) and integrating gives

$$f_R(r) = \frac{6}{\sqrt{\pi}} \exp\left(\tfrac{-r^2}{4}\right) \left[\Phi\left(\frac{r}{\sqrt{6}}\right) - \frac{1}{2}\right], \tag{10}$$

which is the pdf for the range when the sample size is 3. Figure 1 shows a plot of the density function for n=3, and compares it to that for n=2.

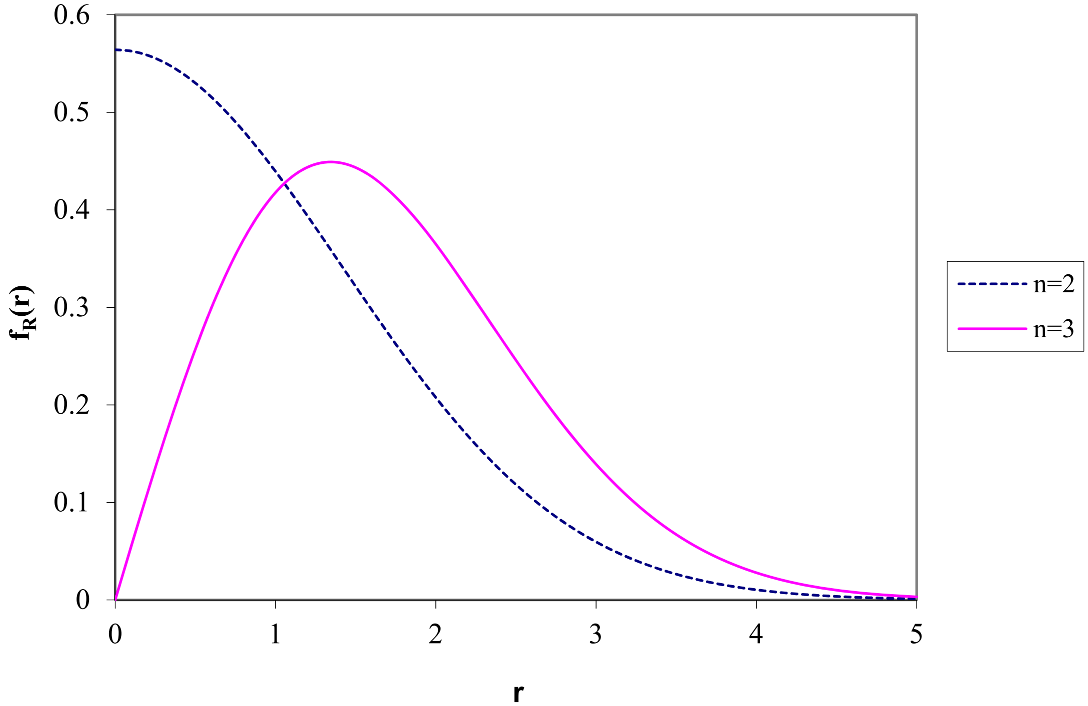
Inserting (10) into (2) and (3) and integrating by parts, we find that $E(R) = d_2 = \frac{3}{\sqrt{\pi}}$ and $d_3 = \sqrt{\frac{2\pi + 3\sqrt{3} - 9}{\pi}}$.

APPROXIMATIONS TO d_2 AND d_3 FOR SAMPLES SIZES OF AT LEAST 4

As mentioned in the previous section, we are not familiar with analytic solutions to expressions (1)-(3) when the sample size exceeds 3; numerical values of the control chart constants d_2 and d_3 are widely available, however. We used the values given in [4, pp. 17, 262-263] to develop rational function expressions to estimate the values of d_2 and d_3. In this section we describe the method used to develop the approximations, and report the results.

[1] In fact, Harter and Balakrishnan [4, pp. 17, 262-263] state that "For $n \geq 3$, it is necessary to resort to numerical integration [to solve (2) and (3)]."

FIGURE 1
Probability density functions of the range of standard normal random variables for sample sizes of 2 and 3



Approximation method

In this section we concentrate on approximating d_2. The method was essentially the same when finding d_3. We began by plotting the tabulated values of d_2 as a function of the sample size n for values of n from 2 to 1000, as given in [6, pp. 45, 189] (we also used values from [4, pp. 17, 262-263] for d_3). Figure 2 shows the plot, which suggests that d_2 is a function of the logarithm of n. Hence, we sought an approximation relating the control chart constants to the logarithm of n.

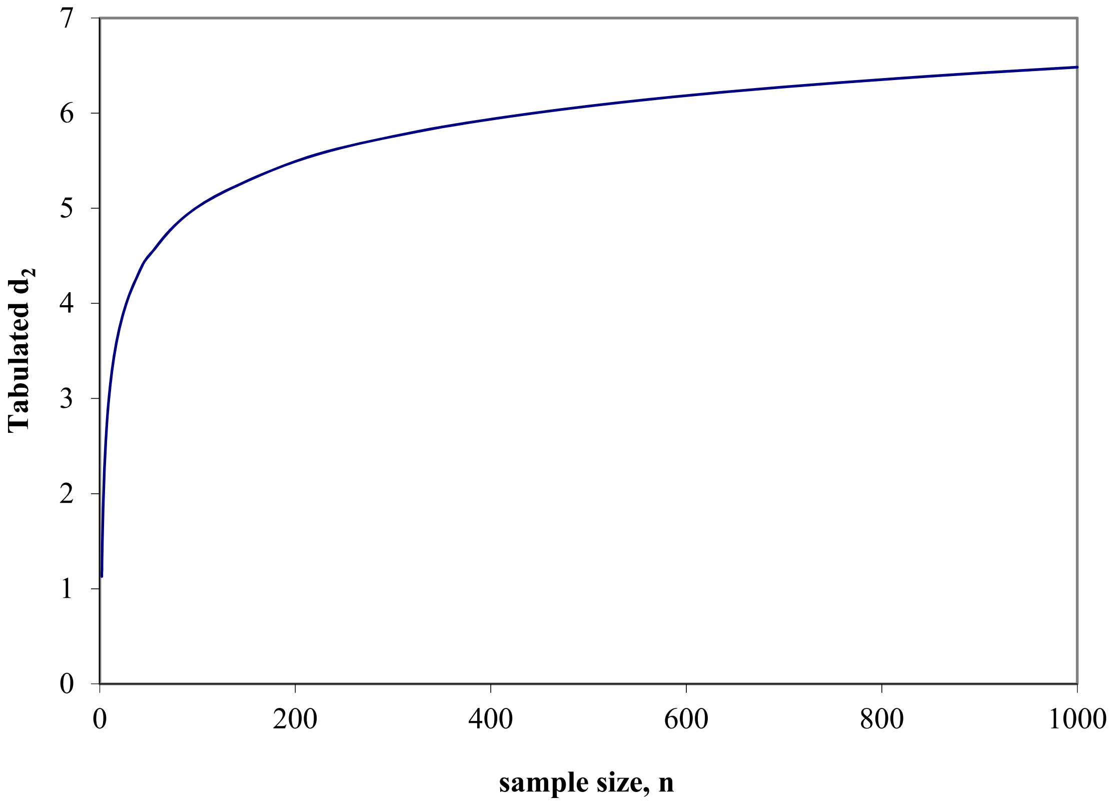
Instead of using a global measure of approximation error such as mean square error, we used as our objective the minimization of the maximum error between the approximations and the tabulated values of the control chart constants. Because d_2 is defined on a discrete point set rather than on an interval, we used the differential correction method to find Chebyshev minimax approximations [7, pp. 311-320]. The algorithm is an iterative one that seeks to minimize the maximum difference between the rational function, $R_{mk}(n)$ and the tabulated values of d_2, where

$$R_{mk}(n) = \frac{\sum_{j=0}^{m} a_j (\log(n))^j}{\sum_{j=0}^{k} b_j (\log(n))^j}, \text{ with } b_0 = 1. \quad (11)$$

In (11), m is the order of the polynomial in the numerator of the rational function, k is the order of the denominator, and the a_j and b_j are coefficients of the numerator and denominator polynomials.

FIGURE 2

Plot of the tabulated values of d_2 as a function of the sample size n.



To begin the procedure, an initial rational function estimate was needed. We used multiple (polynomial) regression to find an initial polynomial solution, and then let the algorithm convert the polynomial expression into a rational one. The regression solutions themselves were quite accurate, but in all cases the solutions were improved by using the differential correction method.

The function $R_{mk}(n)$ has two parameters, m and k, that represent the order of the polynomials in the numerator and denominator of the rational function. These parameters can be adjusted to seek better approximations. It is generally true that increasing $m+k$ improves the precision of the estimates. Hence we tried different combinations of the parameters to find accurate and parsimonious approximations. For d_2 we tried $m+k$ values from 2 to 8, and for d_3 we used values from 4 to 9. Combinations with sums greater than 8 (or 9 for d_3) did not yield significant improvements in accuracy and were discarded in the interest of parsimony. Combinations with sums less than 2 (4 for d_3) gave maximum errors that were unacceptable.

We note that in order to make the computations, not all tabulated values were used. First we used the analytical values for sample sizes 2 and 3. Then for d_2 we used tabulated values for n=4 (1) 25, 30 (10) 50,100 (100) 1000. For d_3 we used tabulated values for n=4 (1) 25, 30 (5) 60, 70 (10) 100, 200, 500, 1000. Removing some of the intermediate values where the curve in Figure 2 is relatively flat resulted in a more manageable optimization problem.

Approximation results

Table 1 summarizes the results for the d_2 approximations, and Table 2 does the same for the d_3 approximations. The tables show the values of m and k, the maximum error for the given rational function, and the coefficient values defined in (11). The tables are sorted by the number of parameters (i.e., by m + k). In the interest of space, the tables only show the case for a given m+k for which the maximum error is smallest. For example, the first entry in Table 1 is for a rational function having m=7 and k=2. The maximum error for such a case was 4.3685x10^{-6}, and the rational function approximation was

$$R_{72}(\log(n)) = \frac{-4.683\text{x}10^{-5} + 4.1602n' - 1.0722n'^2 + .2863n'^3 - .0791n'^4 + .0129n'^5 - .0012n'^6 + 4.363\text{x}10^{-5}n'^7}{1 + .1108n' - .0363n'^2},$$

where $n' = \log(n)$.

Table 1 shows that the best $m + k$ =8 cases was very comparable in terms of their maximum error. (Actually cases for $m + k$ =8 were also comparable, but are not shown in the table to keep the table size manageable.) The table also shows that there was no one systematically preferred form of the rational function approximations. For example, when $m + k$ =8, the best rational function approximation had more terms in the denominator (m =2 and k =6) than in the numerator; when $m + k$ =7, however, the lowest error came from a rational function with more terms in the numerator (m =5 and k =2). Finally, the table shows that the d_2 terms can be accurately estimated to at least 3 decimal places (maximum errors on the order of 10^{-4}) with rational functions having as few as 5 total terms ($m + k$ =4). The approximations are very good with only 8 total terms, having maximum errors of the order of 10^{-6}. Hence very accurate estimates of d_2 can be made for sample sizes from 2 to 1000 using simple algebraic expressions.

Similar results were found for d_3, although the approximations were not quite as good. In order to achieve maximum errors of the order of 10^{-6}, $m + k$ had to be as high as 10. With a total of 8 terms ($m + k$ =7), some maximum errors were just over 10^{-5}, which is still very good. Given the speed of modern computing technologies, rational functions with fewer than 10 terms can be evaluated quickly; hence we believe that the greater accuracy is worth sacrificing a small degree of parsimony. We suggest that the rational functions shown in Tables 1 and 2 with the smallest maximum error be used, as they provide very good accuracy with reasonably-sized rational functions.

SUMMARY

We have provided algebraic expressions that can be used to easily compute the control chart constants d_2 and d_3 for sample sizes from 2 to 1000. For the cases of n=2 and 3, we have provided exact solutions, whereas for cases of $n > 3$ we have provided rational function approximations that are very accurate. Other constants that are based on range control charts (e.g., A_2, D_3 and D_4) can easily be computed from the two constants d_2 and d_3 (see [8 pp. 95, 143] for example). Such expressions allow control chart designers and users, including control chart software producers, to compute the control chart constants without using lookup routines, which generally find control chart constants for a limited number of possible sample sizes. Hence, the expressions simplify the process of automating range control chart construction.

TABLE 1

Rational function values for approximating d_2 (see (11) for definitions of the terms used in the table).

$m + k$	9	8	7	6	5	4	3	2
(m, k)	(7,2)	(2,6)	(5,2)	(2,4)	(1,4)	(1,3)	(2,1)	(1,1)
Maximum Error	4.3685×10^{-6}	4.3907×10^{-6}	4.5173×10^{-6}	2.1767×10^{-5}	7.0438×10^{-5}	1.6052×10^{-4}	5.6083×10^{-4}	2.1880×10^{-2}
a_0	-4.6830×10^{-5}	0.00021168	0.00039446	0.0740863	-0.0021528	-0.0058338	-0.0023213	0.12951861
a_1	4.16017058	4.15771399	4.15608081	3.50035523	4.17780958	4.19907408	4.18711419	3.66813329
a_2	-1.072159	0.34914757	1.4690251	133.771848			0.34490634	
a_3	0.28628465		0.23910449					
a_4	-0.0791025		-0.0147213					
a_5	0.01287818		0.00070213					
a_6	-0.0011768							
a_7	4.36 x10-5							
b_1	0.11084091	0.45026527	0.71813046	31.970664	0.38128612	0.3919365	0.47209716	0.24108419
b_2	-0.036305	0.02450851	0.18294963	12.6518404	-0.0240701	-0.0310113		
b_3		-0.0137816		-1.0557861	-0.0006661	0.00168837		
b_4		0.00412501		0.06259679	0.00029336			
b_5		-0.0006383						
b_6		4.1793×10^{-5}						

TABLE 2
Rational function values for approximating d_3 (see (11) for definitions of the terms used in the table).

$m + k$	10	9	8	7	6	5	4
(*m*, *k*)	(9,1)	(2,7)	(1,7)	(4,3)	(4,2)	(2,3)	(2,2)
Maximum Error	1.0426×10^{-6}	1.4861×10^{-5}	2.6339×10^{-5}	3.9049×10^{-5}	1.1010×10^{-4}	2.0261×10^{-4}	2.1340×10^{-4}
a_0	0.315104	0.222874	-0.22202	0.414579	0.342748	0.457381	0.452862
a_1	5.547612	7.435075	16.809	3.718122	4.964747	3.232791	3.338977
a_2	-2.84442	-1.74755		-0.92875	-0.67509	0.559042	0.491674
a_3	-0.78335			-0.07681	0.305956		
a_4	2.852878			-0.00913	-0.03384		
a_5	-2.51086						
a_6	1.234777						
a_7	-0.36344						
a_8	0.059829						
a_9	-0.00424						
b_1	3.396075	5.437382	15.37598	1.802498	2.973567	1.648605	1.737353
b_2		0.138415	-2.99882	1.976001	2.179108	2.65881	2.63829
b_3		2.584673	14.3693	-0.86642		0.024817	
b_4		-2.56493	-10.1252				
b_5		1.00801	3.946838				
b_6		-0.20567	-0.83038				
b_7		0.017412	0.072844				

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