On A Special Decision Problem

Maryam Hamidi, Department of Industrial Engineering, Lamar University, Beaumont, TX 77710, USA *mhamidi@lamar.edu*
Ferenc Szidarovszky, Department of Mathematics, Corvinus University of Budapest, H-1093, Fővám tér 8, Budapest, Hungary, *szidarka@gmail.com*
Akio Matsumoto, Department of Economics, Chuo University, 742-1, Higashi-Nakano, Hachioji, Tokyo, 192-0393, Japan, *akiom@tamacc.chuo-u.ac.jp*

ABSTRACT

This paper introduces a special problem with two decision makers and real numbers as decision variables. The uniqueness of response functions are guaranteed by the assumption that each profit function is strictly concave in its own decision variable. The existence can be guaranteed by several alternative conditions. Special dynamic extensions are analyzed when the decision makers are not in equilibrium states but try to reach equilibrium with a dynamic process. Three such models are considered: under discrete time scales, under continuous time scales without and with delayed information.

Keywords: Decision making, game theory, delayed information.

INTRODUCTION

Consider a decision problem in which two decision makers (DM) are involved and the profit of each DM depends on the actions of both of them. Such situations are called as two-person games. The fundamentals of game theory can be found in many textbooks and monographs. See for example, Matsumoto and Szidarovszky (2016). For the sake of simplicity assume that their actions are characterized by real numbers x_1 and x_2. Assume that the profit functions $\varphi_1(x_1, x_2)$ and $\varphi_2(x_1, x_2)$ are continuously differentiable on R_+^2 , φ_1 is strictly concave in x_1 with any fixed value of x_2, and similarly, φ_2 is strictly concave in x_2 with any fixed value of x_1. Therefore, with any value of x_2, and DM_1 has a unique maximizer of φ_1, $R_1(x_2)$, and with any value of x_1, DM_2 has a unique maximizer of φ_2, $R_2(x_1)$. These functions are usually called the response functions. The equilibrium of this situation is a pair $(\overline{x_1}, \overline{x_2})$ of decisions of DMs such that: $\overline{x_1} = R_1(\overline{x_2})$ and $\overline{x_2} = R_2(\overline{x_1})$, meaning that the equilibrium decisions are optimal for both DMs assuming that the other DM selects equilibrium decision. Assume that at any time $t > 0$ their decisions are not on equilibrium levels, then they try to use a dynamic process which could lead to equilibrium for both DMs.

DISCRETE ANALYSIS

In the case of discrete time scales the DMs make changes at time periods $t = 0, 1, 2, \ldots$ such that at each time period they try to approach their best responses:

$$x_1(t+1) = x_1(t) + \alpha(R_1(x_2(t)) - x_1(t)) \quad (1)$$
$$x_2(t+1) = x_2(t) + \beta(R_2(x_1(t)) - x_2(t)) \quad (2)$$

where α, β are positive coefficients. In order to avoid overshooting both are assumed to be less than unity. The convergence of these sequences (or stability) is usually examined by linearization around the equilibrium:

$$\widetilde{x_1}(t+1) = (1-\alpha)\widetilde{x_1}(t) + \alpha R_1'(\overline{x_2})\widetilde{x_2}(t) \tag{3}$$
$$\widetilde{x_2}(t+1) = \beta R_2'(\overline{x_1})\widetilde{x_1}(t) + (1-\beta)\widetilde{x_2}(t), \tag{4}$$

where $\widetilde{x_1}(t)$ and $\widetilde{x_2}(t)$ denote their differences from the equilibrium values: $\widetilde{x_1}(t) = x_1(t) - \overline{x_1}$ and $\widetilde{x_2}(t) = x_2(t) - \overline{x_2}$. So, sequences $\{x_1(t)\}$ and $\{x_2(t)\}$ converge to their equilibrium levels if and only if $\{\widetilde{x_1}(t)\}$ and $\{\widetilde{x_2}(t)\}$ converge to zero.

In order to see if these sequences converge to zero, we need to find the eigenvalues of the system by assuming exponential solutions $\widetilde{x_1}(t) = \lambda^t u$ and $\widetilde{x_2}(t) = \lambda^t v$. Substitution of these solutions into equations (3) and (4) gives:

$$\lambda^{t+1}u = (1-\alpha)\lambda^t u + \alpha R_1'(\overline{x_2})\lambda^t v \tag{5}$$
$$\lambda^{t+1}v = \beta R_2'(\overline{x_1})\lambda^t u + (1-\beta)\lambda^t v \tag{6}$$

After simplifying with λ^t, a homogeneous linear algebraic system is obtained for u and v, which has nonzero solutions if and only if its determinant is zero:

$$0 = \det\begin{pmatrix} 1-\alpha-\lambda & \alpha R_1'(\overline{x_2}) \\ \beta R_2'(\overline{x_1}) & 1-\beta-\lambda \end{pmatrix} = \lambda^2 - \lambda(2-\alpha-\beta) + (1-\alpha)(1-\beta) - \alpha\beta R_1'(\overline{x_2})R_2'(\overline{x_1}). \tag{7}$$

It is a quadradic polynomial $\lambda^2 + p\lambda + q$ with $p = -(2-\alpha-\beta)$ and $q = 1-\alpha-\beta+\alpha\beta(1 - R_1'(\overline{x_1})\, R_2'(\overline{x_1}))$. Sequences $\{x_1(t)\}$ and $\{x_2(t)\}$ converge to the equilibrium levels if and only if for all eigenvalues , $|\lambda| < 1$, which is the case if and only if (see Bischi et al., 2010)

$$\begin{gathered} 1+p+q > 0 \\ 1-p+q > 0 \\ q < 1 \end{gathered} \tag{8}$$

In our case the convergence conditions are as follows:

$$\begin{gathered} R_1'(\overline{x_2})R_2'(\overline{x_1}) < 1 \\ R_1'(\overline{x_2})R_2'(\overline{x_1}) < 1 + \frac{4-2\alpha-2\beta}{\alpha\beta} \end{gathered} \tag{9}$$

and

$$R_1'(\overline{x_2})R_2'(\overline{x_1}) > 1 - \frac{\alpha+\beta}{\alpha\beta} \tag{10}$$

Since both α and β are less than unity, the second term in the right-hand side of the second inequality is positive implying that the first inequality is stronger than the second one. Therefore, the sufficient and necessary condition for convergence is the following:

$$1 - \frac{\alpha + \beta}{\alpha\beta} < R_1'(\overline{x_2})R_2'(\overline{x_1}) \quad < 1 \tag{11}$$

CONTINUOUS ANALYSIS

If the time scale is the entire interval $[0, \infty)$, then with any $t \geq 0$ there is no "next time period", so the directions of changes in decisions are modeled. Similarly, to the discrete case both DMs try to move into the direction toward their best responses:

$$\dot{x_1}(t) = \alpha(R_1(x_2(t)) - x_1(t)) \tag{12}$$
$$\dot{x_2}(t) = \beta(R_2(x_1(t)) - x_2(t)) \tag{13}$$

Linearization of the right-hand side around the equilibrium leads to the equations:

$$\dot{\widetilde{x_1}}(t) = -\alpha\widetilde{x_1}(t) + \alpha R_1'(\overline{x_2})\widetilde{x_2}(t) \tag{14}$$
$$\dot{\widetilde{x_2}}(t) = \beta R_2'(\overline{x_1})\widetilde{x_1}(t) - \beta\widetilde{x_2}(t) \tag{15}$$

Similarly, to the discrete case we assume again exponential solutions $\widetilde{x_1}(t) = e^{\lambda t}u$ and $\widetilde{x_2}(t) = e^{\lambda t}v$ leading to the following:

$$\lambda e^{\lambda t}u \quad = -\alpha e^{\lambda t}u + \alpha R_1'(\overline{x_2})e^{\lambda t}v \tag{16}$$
$$\lambda e^{\lambda t}v \quad = \beta R_2'(\overline{x_1})e^{\lambda t}u - \beta e^{\lambda t}v. \tag{17}$$

After simplifying with $e^{\lambda t}$, a homogenous linear algebraic system is obtained, which has nonzero solutions if and only if its determinant is zero:

$$0 = \det\begin{pmatrix} -\alpha - \lambda & \alpha R_1'(\overline{x_2}) \\ \beta R_2'(\overline{x_1}) & -\beta - \lambda \end{pmatrix} = \lambda^2 + \lambda(\alpha + \beta) + \alpha\beta\big(1 - R_1'(\overline{x_2})R_2'(\overline{x_1})\big). \tag{18}$$

It is again a quadratic polynomial $\lambda^2 + p\lambda + q$ with $p = \alpha + \beta$ and $q = \alpha\beta(1 - R_1'(\overline{x_2})R_2'(\overline{x_1}))$. As $t \to \infty$, functions $x_1(t)$ and $x_2(t)$ converge to their equilibrium levels if and only if the real parts of all eigenvalues are negative, which is the case if and only if both p and q are positive (see Bischi at al., 2010). Therefore, the sufficient and necessary condition for convergence is the following:

$$R_1'(\overline{x_2})R_2'(\overline{x_1}) \quad < 1. \tag{19}$$

Comparing conditions (11) and (19) it is clear that convergence in the discrete case implies the same for continuous case, however, convergence in the continuous case does not necessarily imply the same for the discrete case.

ANALYSIS WITH DELAYED INFORMATION

Assume now that the DMs may have access to only delayed information about the decisions of the others. In the discrete case the delay is integer, resulting in a higher order difference equation system. The continuous case is more interesting. Let τ_1 and τ_2 denote the length of delays, then system (12), (13) is modified as:

$$\dot{x_1}(t) = \alpha(R_1(x_2(t-\tau_2)) - x_1(t)) \tag{20}$$
$$\dot{x_2}(t) = \beta(R_2(x_1(t-\tau_1)) - x_2(t)) \tag{21}$$

with linearized version,

$$\dot{\widetilde{x_1}}(t) = \alpha R_1'(\overline{x_2})\widetilde{x_2}(t-\tau_2) - \alpha\widetilde{x_1}(t) \tag{22}$$
$$\dot{\widetilde{x_2}}(t) = \beta R_2'(\overline{x_1})\widetilde{x_1}(t-\tau_1\ ) - \beta\widetilde{x_2}(t). \tag{23}$$

The eigenvalues of this system depend on the lengths of the delays. We now proceed similarly to the previous, no-delay case.

Exponential solutions are again assumed; $\widetilde{x_1}(t) = e^{\lambda t}u$ and $\widetilde{x_2}(t) = e^{\lambda t}v$; which are substituted into the equations to get the following:

$$\lambda e^{\lambda t}u \quad = -\alpha e^{\lambda t}u\ + \alpha R_1'(\overline{x_2})e^{\lambda(t-\tau_2)}v \tag{24}$$
$$\lambda e^{\lambda t}v \quad = \beta R_2'(\overline{x_1})e^{\lambda(t-\ \tau_1)}u - \beta e^{\lambda t}v \tag{25}$$

After simplifying with $e^{\lambda t}$, the determinant of the system becomes:

$$\begin{aligned} 0 = \det&\begin{pmatrix} -\alpha-\lambda & \alpha R_1'(\overline{x_2})e^{-\lambda\tau_2} \\ \beta R_2'(\overline{x_1})e^{-\lambda\tau_1} & -\beta-\lambda \end{pmatrix} \\ &= \lambda^2 + \lambda(\alpha+\beta) + \alpha\beta\left(1 - R_1'(\overline{x_2})R_2'(\overline{x_1})e^{-\lambda(\tau_1+\tau_2)}\right). \end{aligned} \tag{26}$$

Notice first that without delays $\tau_1 = \tau_2=0$, this equation reduces to (18). It is also interesting to mention that the eigenvalues do not depend on the individual delays, only on their sum.

Assume that without delays $x_1(t)$ and $x_2(t)$ converge to the equilibrium levels, that is (19) holds. If the sum $\tau = \tau_1 + \tau_2$ increases, then this convergence property might not hold anymore. The smallest value τ_0 of the sum, when the convergence property is lost is called the critical value. It is also well-known (Matsumoto and Szidarovszky, 2018) that at $\tau=\tau_0$, at least one eigenvalue has zero real part, $\lambda = iw$. Since complex conjugate of an eigenvalue is also an eigenvalue, we may assume that $w > 0$. Substituting this special eigenvalue into equation (26) we have:

$$-w^2 + iw(\alpha+\beta) + \alpha\beta(1 - R_1'(\overline{x_2})R_2'(\overline{x_1}))(\cos w\tau - i\sin w\tau)) = 0 \tag{27}$$

Separating the real and imagining parts

$$\alpha\beta R_1'(\overline{x_2})R_2'(\overline{x_1})\cos w\tau = -w^2 + \alpha\beta \tag{28}$$
$$\alpha\beta R_1'(\overline{x_2})R_2'(\overline{x_1})\sin w\tau = -w(\alpha+\beta) \tag{29}$$

Adding up the squares of these equations:

$$\alpha^2\beta^2(R_1'(\overline{x_2})R_2'(\overline{x_1}))^2 = w^4 - 2w^2\alpha\beta + \alpha^2\beta^2 + w^2(\alpha^2+\beta^2 + 2\alpha\beta) \tag{30}$$

or

$$w^4 + w^2(\alpha^2 + \beta^2) + \alpha^2\beta^2(1 - (R_1'(\overline{x_2})R_2'(\overline{x_1}))^2) = 0 \tag{31}$$

In case, when (19) holds, the constant term is positive showing that there is no positive solution for w^2. Therefore, the convergence property holds for all positive delays, since there is no stability switch.

Assume next that (19) is violated with strict inequality, that is, $R_1'(\overline{x_2})R_2'(\overline{x_1}) > 1$. In this case equation (31) has two real roots for w^2:

$$w_\pm^2 = \frac{-(\alpha^2+\beta^2) \pm \sqrt{D}}{2} \tag{32}$$

with

$$D = (\alpha^2+\beta^2)^2 - 4\alpha^2\beta^2(1 - (R_1'(\overline{x_2})R_2'(\overline{x_1}))^2) > (\alpha^2+\beta^2)^2 \tag{33}$$

implying that $w_+^2 > 0$ and $w_-^2 < 0$. Therefore, we have a unique positive value, $w = w_+$. For the sake of simplicity assume that both response functions are strictly increasing or decreasing, that is,

$$R_1'(\overline{x_2})R_2'(\overline{x_1}) > 0 \tag{34}$$

The other case can be discussed similarly.

From equation (29) it is clear that $\sin w\tau < 0$. The sign of $\cos w\tau$ depends on the value of w^2:

$$\cos w\tau = \begin{cases} > 0 & \text{if } w^2 < \alpha\beta \\ = 0 & \text{if } w^2 = \alpha\beta \\ < 0 & \text{if } w^2 > \alpha\beta \end{cases} \tag{35}$$

so

$$\tilde{\tau}_0 = \begin{cases} (1/w)(2\pi - \cos^{-1}(A)) & \text{if } w^2 \neq \alpha\beta \\ 3\pi/2w & \text{if } w^2 = \alpha\beta \end{cases} \tag{36}$$

where

$$A = \frac{-w^2 + \alpha\beta}{\alpha\beta R_1'(\overline{x_2})R_2'(\overline{x_1})}. \tag{37}$$

The directions of the stability switches are determined by Hopt bifurcation. Select τ as the bifurcation parameter and consider the eigenvalues as functions of τ: $\lambda = \lambda(\tau)$. Implicitly differentiating equation (26) with respect to τ we have:

$$\begin{aligned} 0 &= 2\lambda\lambda' + (\alpha + \beta)\lambda' - \alpha\beta R_1'(\overline{x_2})R_2'(\overline{x_1})e^{-\lambda\tau}(-\lambda'\tau - \lambda) \\ &= 2\lambda\lambda' + (\alpha + \beta)\lambda' - (\lambda^2 + \lambda(\alpha + \beta) + \alpha\beta)(-\lambda'\tau - \lambda) \end{aligned} \tag{38}$$

where we used equation (26) again. So,

$$\lambda' \left(2\lambda + \alpha + \beta + \tau(\lambda^2 + \lambda(\alpha + \beta) + \alpha\beta)\right) + \lambda(\lambda^2 + \lambda(\alpha + \beta) + \alpha\beta) = 0 \tag{39}$$

implying that

$$\frac{1}{\lambda'} = -\frac{2\lambda + \alpha + \beta}{\lambda(\lambda^2 + \lambda(\alpha + \beta) + \alpha\beta)} - \frac{\tau}{\lambda}. \tag{40}$$

The real parts of λ' and $\frac{1}{\lambda'}$ have same sign and at $\lambda = iw$, $\frac{\tau}{\lambda}$ is pure complex number with zero real part, so we need to see the sign of the real part of:

$$\begin{aligned} \frac{-2iw - \alpha - \beta}{iw(-w^2 + iw(\alpha + \beta) + \alpha\beta)} &= -\frac{-2iw - \alpha - \beta}{iw(\alpha\beta - w^2) - w^2(\alpha + \beta)} \\ &= \frac{(-2iw - \alpha - \beta)(-w^2(\alpha + \beta) - iw(\alpha\beta - w^2))}{w^2(\alpha\beta - w^2)^2 + w^4(\alpha + \beta)^2}. \end{aligned} \tag{41}$$

The denominator is positive and the real part of the numerator is

$$-2w^2(\alpha\beta - w^2) + w^2(\alpha + \beta)(\alpha + \beta) = w^2(\alpha^2 + \beta^2 + 2w^2) > 0 \tag{42}$$

showing that at the critical values at least one eigenvalue changes the sign of its real part from negative to positive. That is, convergence cannot return with delayed information.

NUMERICAL SOLUTION

The stable and unstable cases are illustrated. The parameter selection is the following in both cases:

$$b_1 = b_2 = 1\,, \alpha = \beta = 1 \tag{43}$$

with response functions: $R_1(x_2) = -a_1 x_2 + b_1$ and $R_2(x_1) = -a_2 x_1 + b_2$.

In the stable case we have $a_1 = a_2 = 0.9$, in which case $R_1'(x_2) = R_2'(x_1) = -0.9$, so condition (13) is satisfied. The (x_1, x_2) phase diagram is shown in FIGURE 1, where the black curve allows the $\tau_1 = \tau_2 = 0$ case, the blue curve shows the case of $\tau_1 = \tau_2 = 2$ and the red curve corresponds to $\tau_1 = \tau_2 = 3$. The green point is the initial point, and the black point is the equilibrium.

The case of $\tau_1 = \tau_2 = 0$ shows a smooth trajectory, while both delay cases represent oscillatory trajectories.

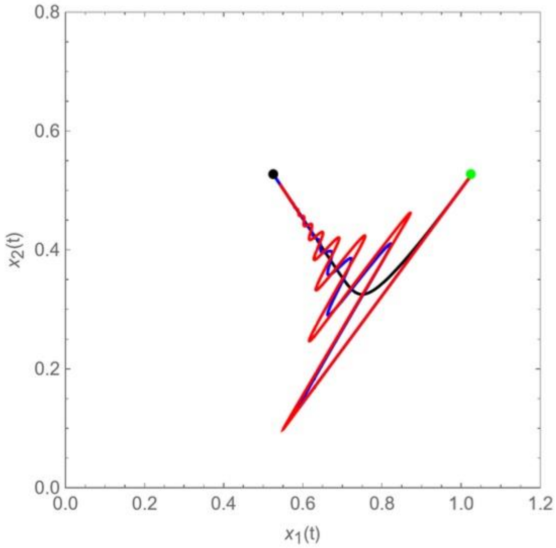

FIGURE 1. Phase diagram in the stable case

FIGURE 2 illustrates the trajectory of $x_1(t)$ with the same color code as before. The convergence in all cases is clearly demonstrated.

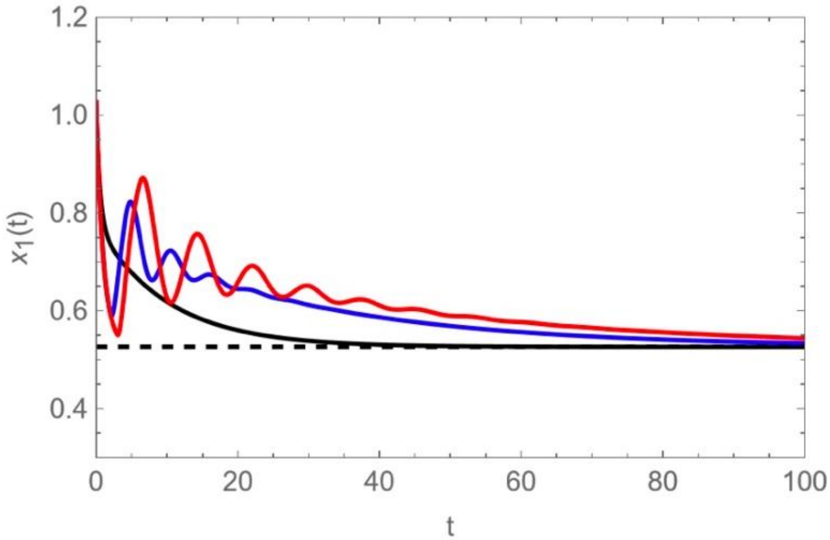

FIGURE 2. Trajectories in the stable case

For the unstable case we selected $a_1 = a_2 = 1.01$, where condition (13) is violated. The time trajectory of $x_1(t)$ is shown in FIGURE 3 with solid lines, the case of $\tau_1 = \tau_2 = 0$ is given in black, the case of $\tau_1 = \tau_2 = 3$ is given in blue and the case of $\tau_1 = \tau_2 = 6$ is given in red color. The trajectory of $x_2(t)$ is illustrated with dotted lines with the same color code as for $x_1(t)$. The divergence of trajectories is clear in all cases.

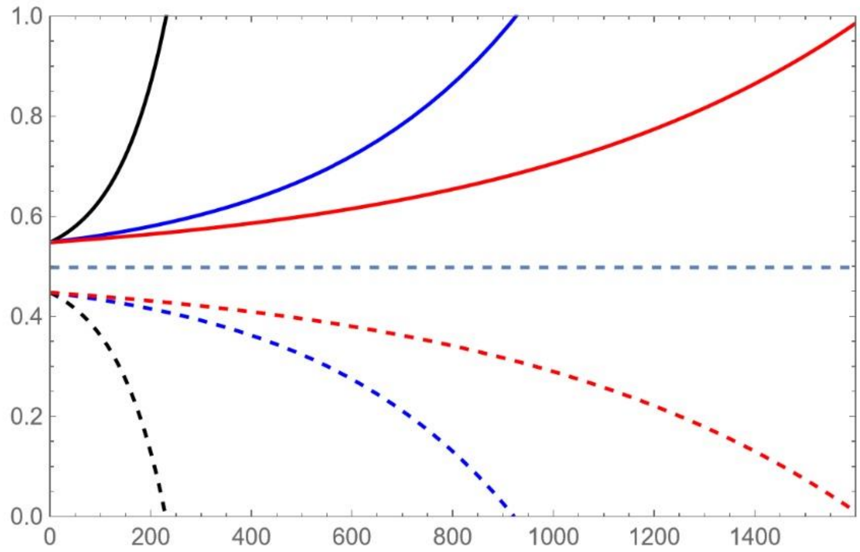

FIGURE 3. Trajectories in the unstable case

CONCLUSIONS

A special problem was analyzed with two decision makers and real numbers as decision variables. The uniqueness of response functions are guaranteed by the assumption that each profit function is strictly concave in its own decision variable. The existence can be guaranteed by several alternative conditions, for example, by the assumption that for all x_2, $\frac{\partial \varphi_1}{\partial x_1}(x_1, x_2)$ has at least one negative value. The assumption for φ_2 is similar. Special dynamic extensions were analyzed when the decision makers are not in equilibrium states but try to reach equilibrium with a dynamic process. Three such models were considered: under discrete time scales, under continuous time scales without and with delayed information. The main results of the paper are as follows:

1. Convergence in the discrete case implies the same in the continuous case.
2. Convergence of the continuous case implies convergence with any positive delays.
3. If there is no convergence in the no-delay case, then it will remain the case with any positive delays.

REFERENCES

Bischi, G-I., C. Chiarella, M. Kopel, F.Szidarovszky (2010). *Nonlinear Oligopolies*. Springer-verlag, Berlin, Heidalberg

Matsumoto, A., F. Szidarovszky (2016). *Game Theory and Its Applications*. Springer Japan KK, Tokyo.

Matsumoto, A., F. Szidarovszky (2018). *Dynamic Oligopolies with Time Delays*. Springer Nature, Singapore.