Multi-Period Pricing and Product Improvement: How Informative Are Product Ratings to New Customers?

Facing the market's mass adoption of product review platforms, sellers should be conscious of the interaction between product ratings and key product decisions including pricing and performance improvement. On the one hand, such decisions should optimally incorporate prior ratings, which are a major source of information on a product's proven performance in the market. On the other hand, they provide an opportunity for a seller to stimulate the ratings which guide future customers' purchase decisions. Employing a multi-period game-theoretic setting, we analyze a forward-looking firm's pricing and product improvement strategies, facing multiple generations of customers who potentially receive a performance signal from prior product ratings. We show that this signal is fully informative in equilibrium only if the product has received a mix of positive and negative reviews. We further establish that the firm increases its optimal price after a period wherein the true performance proves to be greater or slightly less than the ex ante expectation, and decreases the price otherwise. Moreover, when product improvement is cheap, the firm should decrease its improvement size with an increase in the true product performance. By contrast, when the improvement cost is high, the optimal improvement size increases in the product's baseline performance. Finally, we show that the potential for product improvement, even before it is implemented, changes the firm's optimal pricing.

Key words: Product Ratings, Pricing, Product Improvement, Game Theory

1. Introduction

With the increasing accessibility of online review platforms over the past two decades, customers can now effortlessly provide feedback on their consumption experience to manufacturers and service providers as well as to other consumers in the market. This feedback, typically in the form of product ratings, gives the firms a basis for adjusting their prices and determining possible improvements in future product versions. Subsequently, new customers make their purchase decisions upon evaluating the baseline product performance from prior customer ratings and observing the seller's price and product adjustments. The observed price and performance in addition to their unique consumption experience influence the new buyers' contribution to the product ratings, which in turn sway the purchase decisions of later customer generations. In such a market wherein customers can pass along (even though imperfectly at times) their consumption experiences via ratings, it is critical for a company to strategize its pricing and product improvement efforts so as to both integrate the information from prior ratings and steer future ratings to its advantage.

Examples of companies with several product versions, each consumed and rated by its generation of buyers (as explained above), are prevalent in both the manufacturing and service sectors. For instance, iRobot has been producing Roomba[1], an autonomous robotic vacuum cleaner, since

2002 and released its most recent model in 2021. iRobot's customers can relay their consumption experience of each model via their ratings and potentially elaborate reviews on various online marketplaces. The company can use such feedback as a major source of information to price and improve its subsequent models, which are released every one to two years according to the company's website. Other customers also use the ratings and reviews to make their purchase decisions. Typically, those who purchase a Roomba model will use it for several years before upgrading it as the vacuum's primary function will remain the same. Figure 1 illustrates the recent models of Roomba along with the customer ratings. The interaction among product ratings, companies' pricing and product strategies, and customers' buying and rating decisions, which we just explained using the iRobot example, applies to many other manufacturers, for example, of touchless trash cans[2] and point-of-sale (POS) terminals[3]. Typically, a consumer uses a model of such products for several years before upgrading it (skipping many models in between).

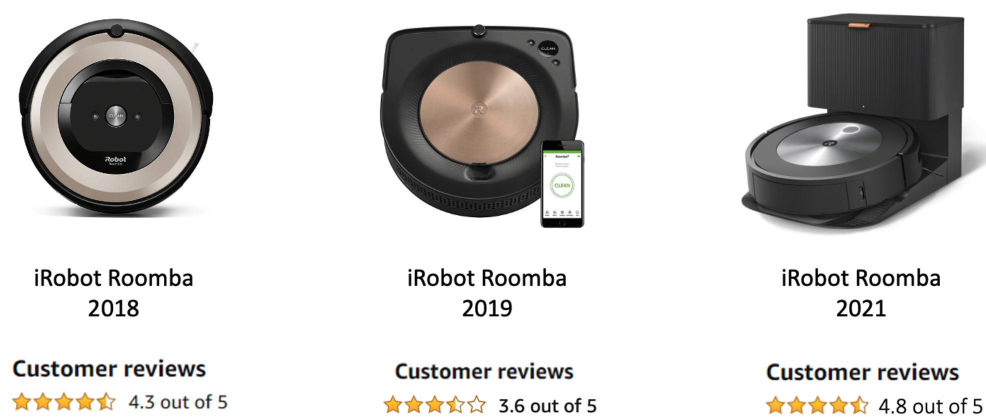

Figure 1 **Illustration of Recent iRobot Roomba Models and Their Customer Ratings**

The service industry also includes examples in which customer ratings and firms' pricing and improvement decisions for consecutive product versions are interdependent. For instance, Microsoft has been offering its Office package (i.e., Office Home & Student or Office Home & Business) since 1995.[4] To buy the newest Office version, which was released in 2021, customers pay a price from $124 to $429, depending on the package they need. Customers can then submit their ratings and reviews on the Microsoft platform or other marketplaces where Office is available for sale.[5] New Office versions, which are offered every two to three years, are designed and priced considering prior customers' feedback. Microsoft stipulates that the users can "only install these versions of Office on one device," and although they "get security updates and fixes during the supported period, [they] won't receive new features."[6] Similar to Microsoft, many professional movie and music production software companies offer versioned applications.[7] For example, Steinberg has been producing Cubase, a popular Digital Audio Workstation (DAW), since 1989 and releasing a new version every two to three years. The users' consumption feedback, predominantly in the form of product reviews and ratings, help Steinberg to adjust the product's features and price across versions. While the users receive security updates and fixes after purchasing a software version, they cannot upgrade to a subsequent version unless they purchase it.[8]

Motivated by the examples above, we investigate the following research questions in this paper: (1) How do customer ratings influence a firm's optimal pricing decision over multiple periods? (2) How do customer ratings influence a firm's optimal decision on product improvement over multiple periods? (3) How does the product improvement capability affect a firm's optimal pricing strategy and ensuing financial metrics? To examine these questions, we consider a monopolist selling a product over multiple periods to consumers who purchase at most one product. We assume that a new group of consumers arrive at the market in each period and provide their ratings at the end of the period based on their consumption experience and perceived price. At the start of each period, the firm observes the customer ratings from the previous period(s), if any, and sets the price in that period. New customers make their purchase decision based on the observed price and all the previous product ratings. To address the effect of the product improvement capability, we consider two scenarios, the first of which does not allow for improvement while the second incorporates the improvement level as the firm's decision variable besides the price. In both scenarios, we characterize the optimal pricing decisions, customer ratings, and firm profits. We then compare the outcomes between the two scenarios to present further managerial insights.

Our results and insights help decision-makers adopt more informed pricing and product improvement strategies, facing customer ratings over multiple periods. Our preliminary findings, which lay the foundation for further results, suggest that customers' unanimously positive evaluation of a product that proves better performance than (initially) expected is not fully informative vis-à-vis its true performance. On the contrary, a less-than-perfect average rating reveals the true product performance (to the firm and next customer generations) in equilibrium. Furthermore, we show that the direction of the seller's price adjustment is *often* consistent with the position of the realized product performance relative to the ex ante expectation. A counterintuitive exception is that, when the realized performance is only *moderately* less than expected, the firm increases its price due to the lack of leverage to signal high performance via high ratings (induced by low prices).

When the seller possesses the product improvement capability, we show that the improvement strategy is dictated by the cost of improvement: At low costs, the seller implements a maximum improvement, which understandably decreases in size as the baseline product performance increases. Interestingly, at high costs, the seller can afford larger improvements with an increase in the baseline performance, which drives higher levels of willingness to pay. Whenever the scenario with improvement yields higher optimal prices than in the no-improvement scenario, the customer ratings are lower (in the former scenario), and vice versa. Additionally, when the improvement cost is cheap (expensive), the average rating in the period preceding an optimal product improvement is inferior (superior) to the average rating in the same period under the no-improvement scenario. This result suggests that customers rate the same product differently based on merely the prospect of product improvement—which only future customers will enjoy. Similarly, the potential of product improvement per se changes the optimal price the firm charges in the first period, that is, even before implementing any improvement.

The remainder of our paper is organized as follows. In §2, we present the relevant literature. We next introduce the model set-up in §3. In §4, we analyze and present the results of the first scenario (i.e., the firm only makes pricing decisions). In §5, we present the second scenario wherein the firm also has a product improvement capability. We compare these two scenarios in §6 to derive more insights. Lastly, we provide concluding remarks in §7.

2. Literature Review

Although a few studies have examined pricing decisions, product upgrades/improvement, and customer reviews altogether, the majority of the relevant literature has mainly focused on only two of these components simultaneously. In this section, we review these research streams which consider pricing and customer ratings, pricing and product upgrades, and lastly, all three together.

In the stream that studies pricing and customer reviews, Sun (2012) analyzes a two-period model wherein a retailer sets period-specific prices, and consumers purchase the product in both periods. She concludes that, due to the informational value of reviews, a higher variance in ratings (when the average is low) corresponds to a higher succeeding demand. Kuksov and Xie (2010) study a retailer's pricing and offering of frills when ratings influence consumers' purchase decision. Via a two-period model, Papanastasiou and Savva (2017) examine a monopolist's pre-announced and responsive pricing given a forward-looking consumer population's reviews. He and Chen (2018) study the pricing strategy of an electronic products retailer when consumers gradually learn the product quality from consecutive customer ratings. They also investigate the effect of information externalities over time on the retailer's optimal pricing. Feng et al. (2019) consider a two-period model wherein consumers' utilities are driven by their idiosyncratic tastes and the exogenous product quality. Zhao et al. (2020) study a framework in which first-period consumers face uncertainty in terms of both product valuation and future price, and use online reviews to communicate their consumption experience to second-period consumers. Stenzel et al. (2020) study a finite-horizon dynamic-pricing model with unknown product quality. They find that, while higher prices are expected to decrease consumers' net utility and worsen the average ratings, they may initially only attract those who perceive a strong fit and improve the ratings. Not immediately close to our work, Jiang and Guo (2015) and Shin et al. (2021) study the effect of different review system designs on pricing decisions. Also, Crapis et al. (2017), Ifrach et al. (2019), Papanastasiou (2020), and Wang et al. (2021) focus on social learning (via reviews) in alternative settings—i.e., with non-Bayesian decisions, two-sided newsvendor models, or multi-level supply chains. Our work differs from and contributes to the reviewed literature stream in at least two major ways: First, we let the firm decide about both its price and product improvement while the reviewed papers consider exogenous or no improvement. Moreover, we study the effect of product improvement on customer reviews and the seller's optimal price and profit. To that end, we consider a three-period model, which is the most parsimonious setting to capture the effect of initial customer ratings on the firm's product improvement and pricing decisions, which in turn influence the subsequent product ratings dictating new consumers' purchase decisions.

We next review the research stream that analyzes pricing in conjunction with product improvement, first focusing on monopolistic settings. Papers in this vein consider a firm that learns the production cost over multiple periods (Teng and Thompson 1996), faces rapid innovations in successive periods (Kornish 2001), offers product return opportunities to customers (Mukhopadhyay and Setaputra 2007), or confronts a service quality variability which customers can learn over time (DeCroix et al. 2021). Similar to these works, our model analyzes a monopolist's joint pricing and product improvement decisions. Unlike these papers though, we examine the interaction of these decisions with customer ratings. Scholars also investigate the pricing of future upgrades for a purchased product. Bala and Carr (2009) study a firm's optimal price given the relationship between the product upgrade and the cost of upgrading complementary pieces. Mehra et al. (2012) analyze a firm's behavior-based price discrimination for product upgrades by considering a price discount to attract existing customers of the competitor. Via a two-period model, Jia et al. (2018) examine inter-temporal, behavior-based, and hybrid price discrimination strategies with an upgrade possibility in the second period. Çakanyıldırım et al. (2020) identify the optimal timing, amount, and pricing of dynamic service upgrades between two-time epochs. Our paper differs from this body of literature as we assume that, consistent with the examples presented in the introduction, the available product version in each period already integrates improvements relative to older versions and will receive version-specific updates at no additional cost. Further, we incorporate customer ratings in the retailer's decision-making, unlike the body of literature that studies pricing and product improvement in alternative supply-chain settings. For instance, Xu (2009) and Chen et al. (2017) consider price and quality decisions for a retailer and a distributor when the retailer can also sell directly to the market. Shulman and Geng (2013) consider two horizontally and vertically differentiated firms with a segment of rational customers who are uninformed of the upgrade fees at the original purchase. Sun et al. (2020) examine an incumbent's optimal software upgrading and pricing strategies while facing competition from an entrant offering software as a service.

Lastly, we discuss papers with all three components—i.e., pricing, product improvement, and customer ratings. Zhao and Zhang (2019) examine the pricing and service quality of a supplier that provides customer-intensive services in an M/M/1 queue and can adjust the service performance across periods. Wang et al. (2019) study a two-period model wherein the retailer chooses the market price, and the manufacturer optimizes its wholesale price and product improvement. While the reviewed papers deem customer ratings as solely a function of product improvement, we assume that ratings are a function of both price and product improvement—consistent with Kuksov and Xie (2010). Besides, our three-period model captures the long-term mutual interactions of customer ratings and such firm decisions as pricing and product improvement. In a related stream, customer ratings perfectly signal the true product performance and do not incorporate consumers' valuation uncertainties. For example, Jiang and Yang (2019) study the role of customer reviews on a retailer's cost efficiency of production and sales. Li et al. (2021) examine the interplay between product design strategies for fast-moving consumer goods and customers' negative reviews along with implications on the competition. Yan and Han (2021) investigate a firm's pricing along with the adoption of a re-manufacturing entry strategy. In the mentioned models, customers have complete information on the product performance upon observing initial reviews, and the firm decides on managing its cost efficiency (Jiang and Yang 2019), product design (Li et al. 2021), or re-manufacturing strategy (Yan and Han 2021). In our model, ratings carry only limited (imperfect) information on product performance, due to the ex ante valuation uncertainties of initial buyers. In summary, we contribute to the literature on firms' pricing and product improvement decisions which both influence and are influenced by customer ratings.

3. Model

We consider a monopolist selling a single product over three periods to consumers with heterogeneous and a priori unknown valuations. We assume each individual consumer to be small relative to the size of the market, which is considered 1 in each period. Each consumer purchases at most one unit of the product, and a new group of consumers arrives at the market in each period. Each consumer i's valuation of the product consists of an idiosyncratic component, denoted by v^i, and a product-specific component, denoted by η. Consumers are heterogeneous (only) with respect to the idiosyncratic valuation component v^i, which is independent and identically distributed (i.i.d.) across consumers, following a uniform distribution on the $[0, V]$ support. Consumers know their idiosyncratic valuation of the product, even before purchase. On the other hand, consumers' valuations are ex ante unknown and ex post homogeneous with respect to the product-specific component, which we also refer to as the product *performance* henceforth. In the absence of prior information, the product-specific valuation component, denoted by random variable H, follows a uniform distribution on $[-1, 1]$, but any prior information on the product—gleaned by product ratings in the context of our paper—potentially narrows down this distribution. Each consumer fully realizes the true product performance, which we denote by η, after consumption. Facing a price of p_t in period $t = 1, 2, 3$, consumer i's valuation is given by

$$U_t^i = v^i + H - p_t, \tag{1}$$

which resolves to $u_t^i = v^i + \eta - p_t$ after consumption. Consistent with practice, consumers arriving in each period directly observe the product price in that period but not the price in other periods. Consumers' purchase decisions are dictated by whether their expectation of the utility in (1) is non-negative ex ante. We assume that a consumer purchases the product in the face of zero utility. Obviously, a customer's realized utility after purchasing the product and perceiving its value (with respect to η) could be greater, equal, or less than the customer's prior expectation.[9]

Customers rate the purchased products based on the realized utilities. We consider a binary rating system according to which customers with positive (or zero) realized utility give the product a rating of 1, and those with negative utility give a rating of 0 in each period. This rating system is consistent with Kuksov and Xie (2010) who argue that customer ratings are a function of net utilities, and accordingly depend on the product price. Later customers receive the average (or equivalently, the percentage of positive) product ratings given by early customers. We denote the average product rating in period t by R_t. Consistent with practice, we assume that customer ratings are the only means through which consumers across different periods communicate. Naturally, first-period consumers make their purchase decision without any access to prior ratings. However, consumers who arrive in the second period can observe and use the ratings from the first period to make deductions about the product performance H. Therefore, the purchase decision in the second period is potentially more informed (than that in the first period) since it incorporates such information that can narrow down the distribution of the product performance ex ante and indirectly affect the ratings second-period customers leave ex post. Consumers who arrive in the third period have access to the average ratings from both the first and the second periods (R_1 and R_2), and can differentiate between the two as well. The latter assumption is motivated by real-world examples in which consumers are able to differentiate between early and later ratings on many online review platforms (e.g., Amazon's, Apple Store's, Yelp's, etc.), which present review dates for different product variants.

At the beginning of each period t, the firm observes the customer ratings from the previous period(s), if any, and sets the price p_t in that period. The firm aims at maximizing its expected profit at the outset of the three-period horizon and is forward-looking; that is, it takes into account the fact that the price of the product in a period affects not only the purchasing decision of the consumers arriving in the same period but also the product ratings they leave, and in turn, the future consumers' purchasing decision which is influenced by these ratings. We assume that the ratings are the only source of information on the product performance—both for the firm and consumers. That is, at the beginning of each period, the firm has the same information set on the product performance (H) as the consumers arriving in that period do. We consider two scenarios for the firm as follows. In the first scenario, the firm only decides on the price in each period. In the second scenario, the firm has an additional lever, which is the decision on the magnitude of product performance improvement in each period. To distinguish the results in the two scenarios, we predominantly use subscript N (signifying "no improvement") for the functions pertaining to the former scenario, and I (standing for "improvement") for those related to the latter scenario.

4. Pricing Decision Analysis

In this section, we focus on a scenario where the firm decides only about the price of the product in each period—leaving out product improvements for now. In every period, starting from the first period moving forward, we characterize the optimal behavior of consumers who observe the price in that period and prior customer ratings, if any. Then we analyze the firm's pricing decision in each period, starting from the third period moving backward.

Upon deciding about their purchase in the first period, consumers do not have any prior information on the intrinsic product performance (H)—although they know their idiosyncratic valuation of the product (v^i). After consuming the product and realizing the product performance, first-period customers leave their ratings, which potentially materialize in a range due the heterogeneity in the idiosyncratic product values (despite the ex post homogeneity of the product performance, η). Second-period customers observe the ratings from the first period, or the equally informative average rating $R_1(\eta, p_1)$ which, for brevity, we sometimes refer to as just R_1. Based on their belief about the first-period price, they form an expectation of the product performance, $\hat{\eta}_{N2}(p_1)$, potentially influencing their purchase decision in the second period. After consuming the product, second-period customers leave their ratings which average out to $R_2\big(\eta, p_2, R_1(\eta, p_1)\big)$ (also referred to as R_2 in the paper), shaping the third-period potential customers' expectation on the product performance, denoted by $\hat{\eta}_{N3}(p_1, p_2)$. This expectation hinges on the third-period consumers' belief about the price charged in the first two periods. Lemma 1 characterizes the first- and second-period consumers' average ratings after consuming the product purchased at p_1 and p_2, respectively. It also characterizes the second- and third-period customers' expectations of the product performance as functions of the prices charged in the previous period(s) as well as the customers' beliefs about the prices, which we denote by $\hat{p}_1$ and $\hat{p}_2$ in the first and second periods, respectively. Note that the beliefs would match the corresponding actual prices in the rational expectation equilibrium.

Lemma 1. *The first- and second-period customers' average ratings are characterized as follows.*

$$R_1(\eta, p_1) = \begin{cases} 1 & \eta \geq 0 \\ 1 + \frac{\eta}{V - p_1} & \eta < 0 \end{cases} \tag{2}$$

$$R_2\big(\eta, p_2, R_1(\eta, p_1)\big) = \begin{cases} 1 & R_1(\eta, p_1) = 1, \eta \geq \frac{1}{2} \\ \frac{V - p_2 + \eta}{V - p_2 + 1/2} & R_1(\eta, p_1) = 1, 0 \leq \eta < \frac{1}{2} \\ 1 & R_1(\eta, p_1) < 1, \eta < 0 \end{cases} \tag{3}$$

Furthermore, the second- and third-period customers' pre-purchase expectations of performance are:

$$\hat{\eta}_{N2}(p_1) \equiv \mathbb{E}[H | \eta, p_1, \hat{p}_1] = \begin{cases} 1/2 & \eta \geq 0 \\ \eta \frac{V - \hat{p}_1}{V - p_1} & \eta < 0 \end{cases} \tag{4}$$

$$\hat{\eta}_{N3}(p_1, p_2) \equiv \mathbb{E}[H | \eta, p_1, \hat{p}_1, p_2, \hat{p}_2] = \begin{cases} 3/4 & \eta \geq 1/2 \\ \frac{V - p_2 + \eta}{V - p_2 + 1/2}(V - \hat{p}_2 + 1/2) - (V - \hat{p}_2) & 0 \leq \eta < 1/2 \\ \eta \frac{V - \hat{p}_1}{V - p_1} & \eta < 0 \end{cases} \tag{5}$$

The specification of the first-period average rating in Lemma (1) maintains that, if the realized performance of the purchased product is at least equal to the ex ante expectation of 0, customers leave a positive rating of 1. Otherwise, only a fraction of customers with high enough valuations (that make up for the lower-than-expected product performance) leave a positive rating. The average rating, in this case, is calculated as the second piece in (2). Consumers arriving at the second period observe the first-period average rating, which, if equal to 1, implies that the product performance is non-negative, ruling out the negative portion of support for H and yielding an updated expectation of 1/2. A first-period rating of less than 1 not only does imply that the product performance is less than 1, but also enables consumers to infer the value of product performance (η) as a function of their belief about the price charged in the first period. Second-period consumers' expectation of the product performance in the mentioned cases is reflected in (4)—which gives the true value of η when $\eta < 0$ and $\hat{p}_1 = p_1$.

Second-period consumers' purchasing decision and rating in (3) are based on their product performance expectation in (4) influenced by the first-period ratings in (2). In case they observe a perfect rating (of 1) from the first period, they leave a positive rating only if their realized performance is greater than 1/2, that is, the updated performance expectation at the beginning of the second period. Otherwise, as long as their realized performance is positive, the fraction of consumers with a better-than-expected realized performance is given by the second piece in (3). We argue that, when the average rating from the first period implies a negative performance, consumers are able to directly infer the product performance value, and in fact, the true value in equilibrium where their belief about the first-period price agrees with the actual price ($\hat{p}_1 = p_1$). In this case, only consumers with a positive net utility purchase the product, producing an average rating of 1 in the last piece of (3). The expected product performance in (5) at the beginning of the third period is developed by the same rationale as in the prior period, though using updated information obtained by the second-period ratings. In equilibrium, ratings are still inconclusive as to the true product performance value only if third-period consumers observe perfect ratings from both periods (implying $\eta \geq 1/2$). Otherwise, the value of the product performance can be solved from the average rating of the second period for positive values, and from the average rating of the first period for the negative values, of the product performance. These cases constitute the second and third pieces of the third-period performance expectation in (5), respectively.

Having characterized the consumer's problem, we start analyzing the firm's problem from the third period backward. As mentioned in §3, market consumption is the only source of information on the product performance, implying an information symmetry (vis-à-vis H) between the firm and consumers. Consequently, the firm's pricing decision in the third period is based on the customers', akin to its own, inference about the product performance—i.e., $\hat{\eta}_{N3}(p_1, p_2)$—gleaned by the ratings from the first two periods. In the third period, any consumer whose expected total valuation, $v^i + \hat{\eta}_{N3}(p_1, p_2)$, exceeds the price ($p_3$) purchases the product. Thus, the firm's problem in the third period is as follows.

$$\max_{p_3} \pi_{N3}(p_1, p_2, p_3) = p_3 \left(\frac{V + \hat{\eta}_{N3}(p_1, p_2) - p_3}{V} \right) \tag{6}$$

In the second period, the firm maximizes its forward-looking expected profit, which we denote by $\tilde{\pi}_{N2}$. This profit function includes the deterministic profit collected in the second period (π_{N2}) plus the third-period expectation of the optimal profit obtained from (5), $\pi^*_{N3}(p_1, p_2)$, over all feasible values of product performance H narrowed down by the information from the first-period ratings. The optimization problem formulation is as follows.

$$\max_{p_2} \tilde{\pi}_{N2}(p_1,p_2) \equiv \pi_{N2}(p_1,p_2) + \mathop{\mathbb{E}}_H[\pi^*_{N3}(p_1,p_2)|R_1] = p_2\Big(\frac{V - p_2 + \hat{\eta}_{N2}(p_1)}{V}\Big) + \mathop{\mathbb{E}}_H[\pi^*_{N3}(p_1,p_2)|R_1] \quad (7)$$

Finally, in the first period, the firm chooses the price p_1 to maximize its forward-looking expected profit $\tilde{\pi}_{N1}$ as presented below. The first term of the profit function represents the deterministic profit in the first period; the second term is the expectation of the forward-looking optimal profit in the second period, which we denote by $\pi^*_{N2}(p_1)$, over all feasible values of H.

$$\max_{p_1} \tilde{\pi}_{N1}(p_1) \equiv \pi_1(p_1) + \mathop{\mathbb{E}}_H[\pi^*_{N2}(p_1)] = p_1\Big(\frac{V - p_1}{V}\Big) + \mathop{\mathbb{E}}_H[\pi^*_{N2}(p_1)] \quad (8)$$

Proposition 1 characterizes the solution to the firm's problem explained above. The optimal prices are obtained by analyzing the third-period problem first, and then moving backward to the analysis of the second period and, eventually, the first period.

Proposition 1 *Consider a firm setting a price for its product at the beginning of each of the three periods, observing customer ratings from prior periods (if any). The optimal prices, i.e., solutions to (6), (7), and (8) are as follows.*

$$p^*_{N1} = \frac{1}{12}(9V - \sqrt{9V(V+4) - 24}) \quad (9)$$

$$p^*_{N2} = \begin{cases} \frac{9+18V-\sqrt{6}\sqrt{6V^2+9V+2}}{24} & \eta \geq 0 \\ \frac{V+\eta}{2} & \eta < 0 \end{cases} \quad (10)$$

$$p^*_{N3} = \begin{cases} \frac{4V+3}{8} & \eta \geq 1/2 \\ \frac{V+\eta}{2} & \eta < 0 \end{cases} \quad (11)$$

Corollary 1 presents the customer ratings in equilibrium at the end of the first and second periods, directly derived by plugging the above optimal prices into the expressions in Lemma 1.

Corollary 1 *For a firm charging optimal prices for the product it sells over a three-period horizon, the customer ratings from the first two periods are as follows.*

$$R^*_{N1} = \begin{cases} 1 & \eta \geq 0 \\ 1 + \frac{12\eta}{3V+\sqrt{9V(4+V)-24}} & \eta < 0 \end{cases} \quad (12)$$

$$R^*_{N2} = \begin{cases} 1 & \eta \geq \frac{1}{2} \vee \eta < 0 \\ 1 + \frac{12(2\eta-1)}{3+6V+\sqrt{6}\sqrt{6V^2+9V+2}} & 0 \leq \eta < \frac{1}{2} \end{cases} \quad (13)$$

Figure 2 provides a graphical representation of the results in Proposition 1 (panel *a*) and Corollary 1 (panel *b*), wherein optimal prices and customer ratings are shown as functions of the realized product performance, η. We first explain the results for $\eta \geq 0$, and then for $\eta < 0$. For any given range of η, we present the insights regarding the customer ratings first, and the firm's optimal pricing strategy afterwards.

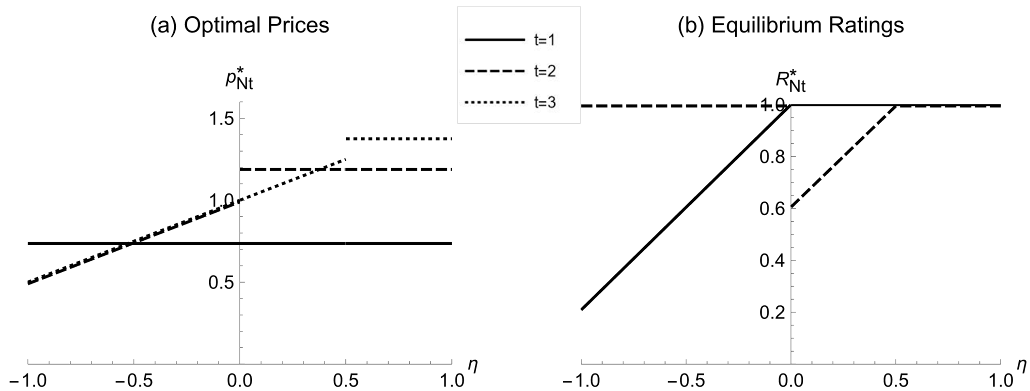

Figure 2 Optimal Prices and Ratings against the True Product Performance ($V = 2$)

To understand the development of ratings and prices in the positive product performance region, note that only the customers with positive expected net utilities, which subsume a zero performance expectation, purchase the product in the first period. At the end of the first period, if the product performance is better than expected (i.e., $\eta \geq 0$), customers gain a positive utility; thus, the average rating at the end of the first period is at its maximum, i.e., $R^*_{N1} = 1$, as shown in Figure 2–*b*. In this case, the only information the customer ratings convey is that the product is better than initially expected. In view of this information, the firm increases the optimal second-period price, but as shown in Figure 2–*a*, this price does not vary with η because the ratings cast no light on the exact value of the product performance. At the beginning of the second period, customers update their expectations of the product performance to a positive value (i.e., $\mathbb{E}[\eta] = 0.5$). Upon consuming the product in the second period, if customers obtain a higher utility than this expectation, they all review the product in a favorable way, resulting in $R^*_{N2} = 1$. As through the formerly explained logic, the ratings do not convey any information (to the firm or future consumers) beyond that the actual product performance is better than the prior expectation, leading to an increase in the optimal price by a fixed margin when $\eta \geq 0.5$. Otherwise, if the product performs worse than expected, the firm receives a less-than-perfect average rating because some customers wind up realizing a negative net utility. Throughout this range of performance, third-period consumers infer the true product performance in equilibrium and thus are willing to pay higher prices for better-performing products signaled through higher ratings from the second period (i.e., increasing ratings in η).

When the product performance is lower than the initial expectation ($\eta < 0$), the behavior of the second-product price mimics that of the third-period price in the previous scenario (with $\eta \geq 0$), due to similar dynamics. Since, in this case, a portion of customers obtains a negative utility, the product receives mixed ratings from which consumers arriving in the second period can derive the exact value of the product performance in equilibrium. Within the negative performance range, with an increase in the true product performance, the portion of such customers (with negative realized utility) decreases, and the average rating in equilibrium (R^*_{N1}) increases (see Figure 2–b), thus improving the potential for charging a higher price in the second period (see Figure 2–a). Because the first-period ratings fully reveal the product performance, the second-period ratings do not render any additional information. Hence, the firm's pricing strategy for the third period follows that in the second period. Since the fully informative ratings at the end of the first period leave no product performance uncertainty in the second-period market, only consumers with positive net utility purchase the product. Therefore, no consumer obtains negative utility after consumption in the second period, yielding $R^*_{N2} = 1$ in the negative η range.

Considering the price dynamics (over time), we observe in Figure 2–a that if the realized product performance is greater than the ex ante expectation in any period, the seller increases the price in the following period. On the other hand, if the realized performance is *substantially* less than the expectation, the firm charges a lower subsequent price. Intuitively, in these two cases, the firm adjusts its price based on the proven performance of the product reflected in the ratings. We can observe a less intuitive result when the average ratings are only *moderately* less than the ex ante expectation, that is, from the first to the second period when η is in the neighborhood of 0 on the left, and from the second to the third period when η is in the neighborhood of 0.5 on the left. Interestingly, the firm increases its optimal price in these cases although the realized performance is less than the ex ante expectation. The intuition for this price increase lies in the particularly low initial price, which we next explain based on the transition from the first to the second period—a similar intuition applies to the transition from the second to the third period in a different η range. Consider the firm's first-period profit-to-go, which consists of the first-period profit plus an expectation of profits in future periods. By setting a low price in the first period (specifically less than what maximizes the first-period profit only), the firm helps boost the initial ratings, stimulating revenues in subsequent periods. Mathematically, the second term in (8) is increasing in customers' pre-purchase expectations of the product performance in the second and third periods, $\hat{\eta}_{N2}(p_1)$ and $\hat{\eta}_{N3}(p_1, p_2)$, both of which are decreasing in p_1. In case the first-period ratings reveal a true performance that is less than initially expected, customers can exactly calculate that performance (in equilibrium), thus thwarting any attempt by the firm to signal a high performance via ratings. Losing the incentive to keep the price low, the firm increases its price in the second period—unless consumption proves a very poor performance, i.e., η is close to -1. In the next section, we present the model with a joint price and product improvement decision.

5. Product Improvement and Pricing Decisions Analysis

We now turn our attention to a scenario in which the firm not only does make a decision about its price in each period but also has the capability of improving its product performance—based on the information gleaned by the customer ratings, if any, from the prior period. In this scenario, in addition to the prices (p_1, p_2, and p_3), the firm also decides about the product improvement levels, which we denote by δ_2 and δ_3 in the second and third periods, respectively. We consider an improvement cost of $c\delta^2$ for the firm when it implements a product improvement of size δ in a given period. Consistent with real-world examples, the cost structure implies that improvements of larger magnitudes are increasingly more costly and helps tractability. We assume that the firm improves the product only when it has a well-founded evaluation of the current product performance and thus an understanding of the gap between the market's perception of the product and the maximum achievable performance—that is, the upper bound of H's distribution on $[-1, 1]$. Consequently, the notion of product improvement at the beginning of the first period (i.e., δ_1) would be superfluous because it would already be integrated into the performance of the product offered to the market at the outset of the purchasing horizon. More information emerging in the second and third periods may provide the firm with the basis for product improvements. As in the baseline scenario, we start by analyzing the firm's decisions in the third period (given its decisions in the first two periods) and work the problem backwards (into the second and first periods, respectively).

Lemma 2 *Consider a firm jointly deciding about its price and product performance.*

(i) The first- and second-period average ratings are as follows.

$$R_1(\eta, p_1) = \begin{cases} 1 & \eta \geq 0 \\ 1 + \frac{\eta}{V - p_1} & \eta < 0 \end{cases} \tag{14}$$

$$R_2\big(\eta, p_2, R_1(\eta, p_1)\big) = \begin{cases} 1 & R_1(\eta, p_1) = 1, \eta \geq \frac{1}{2} \\ \frac{V - p_2 + \eta}{V - p_2 + 1/2} & R_1(\eta, p_1) = 1, 0 \leq \eta < \frac{1}{2} \\ 1 & R_1(\eta, p_1) < 1, \eta < 0 \end{cases} \tag{15}$$

(ii) The second- and third-period customers' pre-purchase expectations of the product performance are characterized as below, respectively.

$$\hat{\eta}_{I2}(p_1,\delta_2)=\begin{cases}1/2 & \eta\geq 0\\ \eta\frac{V-\hat{p}_1}{V-p_1}+\delta_2 & \eta<0\end{cases} \tag{16}$$

$$\hat{\eta}_{I3}(p_1,p_2,\delta_2,\delta_3)=\begin{cases}3/4 & \eta\geq 1/2\\ \frac{V-p_2+\eta}{V-p_2+1/2}(V-\hat{p}_2+1/2)-(V-\hat{p}_2)+\delta_3 & 0\leq\eta<1/2\\ \eta\frac{V-\hat{p}_1}{V-p_1}+\delta_2 & \eta<0\end{cases} \tag{17}$$

The product performance expectations in Lemma 2 mimic those in Lemma 1, except that customers observe and account for the level of product improvement in their expectation of the product performance. The assumption that the product improvement level is observable even before the consumer purchases a product is consistent with many real-world situations[10] in which a firm provides a description on the product upgrade or version update, and helps analytical tractability.

In the third period, the firm solves the following problem, which is similar to that in (6), except for that the firm jointly decides about its price and product improvement level. The market size, captured by the expression in parentheses, incorporates consumers' observation of the firm's product improvement level influencing their purchasing decisions.

$$\max_{p_3,\delta_3}\pi_{I3}(p_1,p_2,p_3,\delta_2,\delta_3)=p_3\left(\frac{V+\hat{\eta}_{I3}(p_1,p_2,\delta_2,\delta_3)-p_3}{V}\right)-c\delta_3^2 \tag{18}$$

Similarly, the structure of the second-period problem in the following resembles that in (7) but additionally incorporates the implication of product improvement on the second-period consumers' expectation of the product performance in (16) as well as the cost associated with product improvement. In the following expression, $\pi^*_{I3}(p_1,p_2,\delta_2)$ denotes the resulting optimal profit from the above maximization problem.

$$\begin{aligned}\max_{p_2,\delta_2}\tilde{\pi}_{I2}(p_1,p_2,\delta_2)&\equiv\pi_{I2}(p_1,p_2,\delta_2)+\mathop{\mathbb{E}}_H[\pi^*_{I3}(p_1,p_2,\delta_2)|R_1]\\&=p_2\Big(\frac{V-p_2+\hat{\eta}_{I2}(p_1,\delta_2)}{V}\Big)-c\delta_2^2+\mathop{\mathbb{E}}_H[\pi^*_{I3}(p_1,p_2,\delta_2)|R_1]\end{aligned} \tag{19}$$

At last, the following first-period forward-looking profit consists of the revenue in the first period plus the expectation of the profit gained in the future periods, which we denote by $\pi^*_{I2}(p_1)$.

$$\max_{p_1}\tilde{\pi}_{I1}(p_1)\equiv\pi_1(p_1)+\mathop{\mathbb{E}}_H[\pi^*_{I2}(p_1)]=p_1\Big(\frac{V-p_1}{V}\Big)+\mathop{\mathbb{E}}_H[\pi^*_{I2}(p_1)] \tag{20}$$

Proposition 2 presents the optimal solution to the firm's problem explained above.

Proposition 2 *When the firm decides about its product improvement levels in addition to the prices, the solution to the firm's problem in (18), (19), and (20) is as follows.*

(i) For $c \leq \frac{1+V}{4V}$*:*

$$
\begin{aligned}
p_{I1}^* &= \frac{3V}{4} - \frac{\sqrt{3V(40c+3V)}}{12} \\
(p_{I2}^*, \delta_2^*) &= \begin{cases} \left(\frac{1}{24}\left(9+18V-\sqrt{9+12V(3+5c+3V)}\right), 0\right) & \eta \geq 0 \\ \left(\frac{1+V}{2}, 1-\eta\right) & \eta < 0 \end{cases} \\
(p_{I3}^*, \delta_3^*) &= \begin{cases} (\frac{V}{2}+\frac{3}{8}, 0) & \eta \geq 1/2 \\ (\frac{1+V}{2}, 1-\eta) & 0 \leq \eta < 1/2 \\ (\frac{1+V}{2}, 0) & \eta < 0 \end{cases}
\end{aligned}
$$

(ii) For $\frac{1+V}{4V} < c < \frac{1+V}{2V}$*:*

$$
\begin{aligned}
p_{I1}^* &= \frac{3V}{4} - \sqrt{\frac{6+3V\left(6-16c^3V^2+2V(3+V)-3c(2+V)(2+3V)+c^2(40V-6V^3)\right)}{12^2cV}} \\
(p_{I2}^*, \delta_2^*) &= \begin{cases} (\frac{3V}{4}+\frac{9}{24}-\sqrt{\frac{6-3V\left(16c^3V^2+3c(4V+3)+8c^2V(6V^2+6V-1)+2(V^2+3V+3)\right)}{24^2cV(1-4cV)}}, 0) & \eta \geq 0 \\ \left(\frac{1+V}{2}, 1-\eta\right) & 1-\frac{1+V}{2cV} \leq \eta < 0 \\ (\frac{cV(V+\eta)}{2cV-1}, \frac{V+\eta}{2cV-1}) & \eta < 1-\frac{1+V}{2cV} \end{cases} \\
(p_{I3}^*, \delta_3^*) &= \begin{cases} \left(\frac{V}{2}+\frac{3}{8}, 0\right) & \eta \geq 1/2 \\ \left(\frac{1+V}{2}, 1-\eta\right) & 1-\frac{1+V}{4cV} \leq \eta < 1/2 \\ \left(\frac{2cV(V+\eta)}{4cV-1}, \frac{V+\eta}{4cV-1}\right) & 0 \leq \eta < 1-\frac{1+V}{4cV} \\ \left(\frac{1+V}{2}, 0\right) & 1-\frac{1+V}{2cV} \leq \eta < 0 \\ \left(\frac{cV(V+\eta)}{2cV-1}, 0\right) & \eta < 1-\frac{1+V}{2cV} \end{cases}
\end{aligned}
$$

(iii) For $c \geq \frac{1+V}{2V}$*:*

$$
\begin{aligned}
p_{I1}^* &= \frac{3V}{4} + \sqrt{\frac{3V\left(2c(3V^2+12V-8)-3V\right)}{12^2(2cV-1)}} \\
(p_{I2}^*, \delta_2^*) &= \begin{cases} \left(\frac{1}{24}\left(9+18V-\sqrt{\frac{24cV(6V^2+9V+2)-36V(V+1)-9}{4cV-1}}\right), 0\right) & \eta \geq 0 \\ \left(\frac{cV(V+\eta)}{2cV-1}, \frac{V+\eta}{2cV-1}\right) & \eta < 0 \end{cases} \\
(p_{I3}^*, \delta_3^*) &= \begin{cases} \left(\frac{V}{2}+\frac{3}{8}, 0\right) & \eta \geq 1/2 \\ \left(\frac{2cV(V+\eta)}{4cV-1}, \frac{V+\eta}{4cV-1}\right) & 0 \leq \eta < 1/2 \\ \left(\frac{cV(V+\eta)}{2cV-1}, 0\right) & \eta < 0 \end{cases}
\end{aligned}
$$

Corollary 2 below provides the ratings in equilibrium at the end of the first and second periods, denoted by $R_{I1}^* \equiv R_{I1}(\eta, p_{I1}^*)$ and $R_{I2}^* \equiv R_{I2}\big(\eta, p_{I2}^*, R_1(\eta, p_{I1}^*)\big)$, respectively. The results are directly obtained by plugging the prices in Proposition 2 into the ratings functions in Lemma 2.

Corollary 2 *At the optimal levels of price and product improvement, customer ratings in the first two periods are as follows.*

$$R^*_{I1} = \begin{cases} 1 & \eta \geq 0 \\ 1 + \frac{\eta}{V - \frac{3V}{4} - \frac{\sqrt{3V(40c+3V)}}{12}} & (\eta < 0) \wedge (c \leq \frac{1+V}{4V}) \\ 1 + \frac{\eta}{V - \frac{3V}{4} - \sqrt{\frac{6+3V\left(6-16c^3V^2+2V(3+V)-3c(2+V)(2+3V)+c^2(40V-6V^3)\right)}{12^2 cV}}} & (\eta < 0) \wedge (\frac{1+V}{4V} < c < \frac{1+V}{2V}) \\ 1 + \frac{\eta}{V - \frac{3V}{4} + \sqrt{\frac{3V\left(2c(3V^2+12V-8)-3V\right)}{12^2(2cV-1)}}} & (\eta < 0) \wedge (c \geq \frac{1+V}{2V}) \end{cases}$$

$$R^*_{I2} = \begin{cases} 1 & (\eta \geq \frac{1}{2}) \vee (\eta < 0) \\ \frac{V - \frac{1}{24}\left(9+18V-\sqrt{9+12V(3+5c+3V)}\right)+\eta}{V - \frac{1}{24}\left(9+18V-\sqrt{9+12V(3+5c+3V)}\right)+1/2} & (0 \leq \eta < \frac{1}{2}) \wedge (c \leq \frac{1+V}{4V}) \\ \frac{V - \frac{3V}{4} + \frac{9}{24} - \sqrt{\frac{6-3V\left(16c^3V^2+3c(4V+3)+8c^2V(6V^2+6V-1)+2(V^2+3V+3)\right)}{24^2 cV(1-4cV)}}+\eta}{V - \frac{3V}{4} + \frac{9}{24} - \sqrt{\frac{6-3V\left(16c^3V^2+3c(4V+3)+8c^2V(6V^2+6V-1)+2(V^2+3V+3)\right)}{24^2 cV(1-4cV)}}+1/2} & (0 \leq \eta < \frac{1}{2}) \wedge (\frac{1+V}{4V} < c < \frac{1+V}{2V}) \\ \frac{V - \frac{1}{24}\left(9+18V-\sqrt{\frac{24cV(6V^2+9V+2)-36V(V+1)-9}{4cV-1}}\right)+\eta}{V - \frac{1}{24}\left(9+18V-\sqrt{\frac{24cV(6V^2+9V+2)-36V(V+1)-9}{4cV-1}}\right)+1/2} & (0 \leq \eta < \frac{1}{2}) \wedge (c \geq \frac{1+V}{2V}) \end{cases}$$

We further discuss and illustrate the results formally stated above in the following two subsections. In §5.1, we investigate the optimal prices, product improvements, profits, and customer ratings over the entire range of the realized product performance (η) for a given level of the unit improvement cost (c). Subsequently, in §5.2, we fix the realized product performance and examine the firm's optimal decisions over a range of the unit improvement cost.

5.1. Results Based on Product Performance for a Given Improvement Cost

We now explain the behavior of the firm's optimal decisions and corresponding market outcomes with respect to the true product performance based on the results in Proposition 2 and Corollary 2.

We begin by discussing the results for small values of c, that is, when product improvement is cheap. In this case, the firm optimally implements the maximum product improvement (i.e., $\delta^*_t = 1 - \eta$) upon observing less-than-perfect customer ratings from the previous period, as should be expected.[11] Referring to Proposition 2–*i*, that occurs when $\eta < 0$ and $0 \leq \eta < 1/2$ at the end of the first and second periods, respectively. As the realized product performance increases, the gap between the current and the maximum possible performance ($\eta = 1$) decreases, leaving less room for product improvement as shown in Figure 3–*a*. The firm will charge a constant price (in η) once it establishes the product performance at its maximum level, i.e., in the second and third periods when $\eta < 0$ and in the third period when $0 \leq \eta < 1/2$ (see Figure 3–*b*). Recall that, when the realized product performance is greater than the ex ante expectations (i.e., in the first period for $\eta \geq 0$ and in the second period for $\eta \geq 1/2$), the firm receives the highest average rating of one, providing no basis for product improvement (see Figure 3–c). Correspondingly, in the given ranges of η, the optimal product improvement is zero. The average customer rating also turns out to be perfect after the firm achieves a maximum performance in the second period for $\eta < 0$.

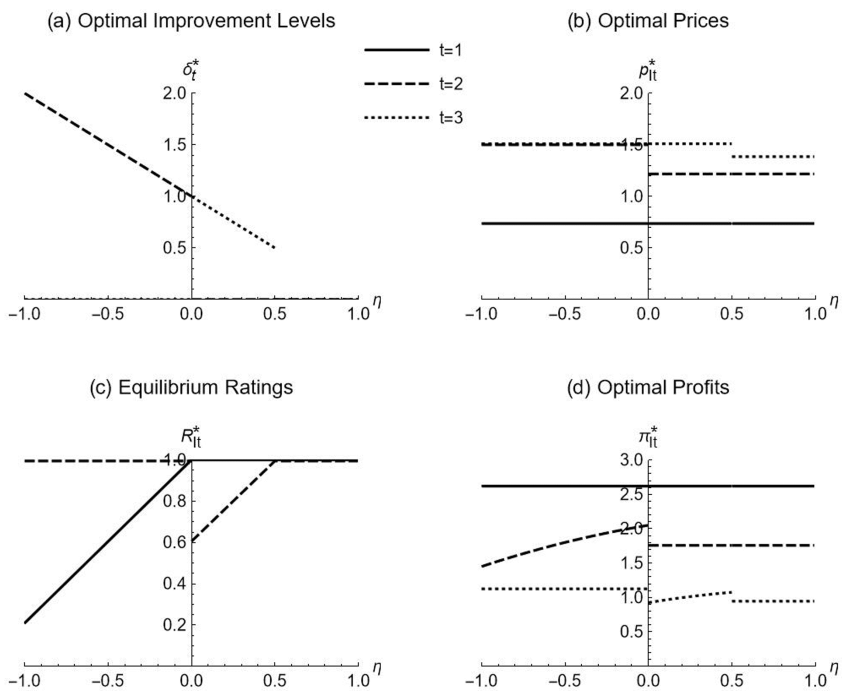

Figure 3 Optimal Improvements, Prices, Ratings, and Profits for Low c Values ($V = 2$, $c = 0.2$)

We now focus on the drivers of price increase over time. At the outset of the time horizon, consumers have no prior information regarding the product performance; thus, the firm sets the first-period price based on the zero performance expectation. The dynamics of price increase (over time) are fairly intuitive when the realized product performance falls short of the ex ante expectation, prompting the firm to apply the utmost performance improvement, i.e., for $\eta < 0$ in the second period and for $0 \leq \eta < 1/2$ in the third period. With the resulting maximum performance, consumers exhibit the highest willingness to pay, which in turn increases the optimal price the firm charges in the given ranges of η. Yet, other dynamics drive the higher price (say, in the second period) when the realized performance (say, in the first period) *exceeds* the ex ante expectation. In this case, the firm applies no product improvement, thus second-period consumers rule out the possibility of inferior performance (relative to $\eta = 0$) and are willing to pay a higher price—though for a different reason from that in the previous case—leading to $p_{I2}^* > p_{I1}^*$ for $\eta \geq 0$ and $p_{I3}^* > p_{I2}^*$ (see Figure 3–b). Note that the expected product performance in these cases, even after receiving the information from prior ratings (or alternatively from the firm's improvement scheme), is less than the maximum performance of $\eta = 1$. As a result, we expect p^*_{I2} and p^*_{I3} in these regions to be less than the corresponding prices after the firm applies a maximum product improvement.

Figure 3–d represents the optimal expected forward-looking profits, which we have denoted by π^*_{It}. As expected, the average profit-to-go decreases over time since the forward-looking profit in each period incorporates an expectation of profits in future periods. We also observe that the expected profit functions are flat through the ranges in which η cannot be inferred, i.e., $\eta \in [-1, 1]$ in the first period, $\eta \in [0, 1]$ in the second period, and $\eta \in [1/2, 1]$ in the third period. In other regions, due to constant prices (in η), profit trends are only influenced by the firm's improvement decision. As anticipated, profits are inversely related to product improvements and thus to the improvement costs, in each period.

When product improvement becomes expensive (i.e., c is high), the firm does not find it optimal to aim for the maximum product performance (unlike in the low-c case). Mathematically, we obtain an interior solution for the optimal product improvement (δ_t) in both the second and third periods, if it occurs at all, that is, when existing customer ratings are informative ($R_{t-1} < 0$). As Figure 4–c shows, customer ratings follow the same pattern as in the previous case (with low c). Interestingly, however, in contrast to the previous case, the firm optimally decides to improve the product more as the average ratings from the prior period increase (see Figure 4–a). To understand this phenomenon, we call attention to the price trends in the corresponding cost ranges, as illustrated in Figure 4–b. As average product ratings increase in these ranges, customers receive the signal of a higher realized product performance and, thus are willing to pay more for the product. In turn, the firm optimally charges a higher price, which provides more leverage to compensate for the high cost of product improvement. Hence, the product improvement magnitude increases with the realized product performance—within the η ranges where customers can infer the true performance, i.e., $\eta < 0$ in the second period and $0 \leq \eta < 1/2$ in the third period. As anticipated, all the profit-to-go functions are non-decreasing in the realized product performance (see Figure 4–d).

We can leverage the elaborated dynamics (among ratings, performance levels, and prices) to explain the observation that, when product improvement is expensive, the firm may reduce its price over time. Figure 4–b shows that such a price drop emerges at $\eta \to -1^+$ from the first to the second period, and at $\eta \to 0^+$ from the second to the third period. In the given neighborhoods of η, the realized product performance is at the lower end of the a priori expected range. Additionally, as explained above, customers do not rationally expect much improvement due to the associated high cost. The realization of inferior performances, together with the anticipation of little improvements,

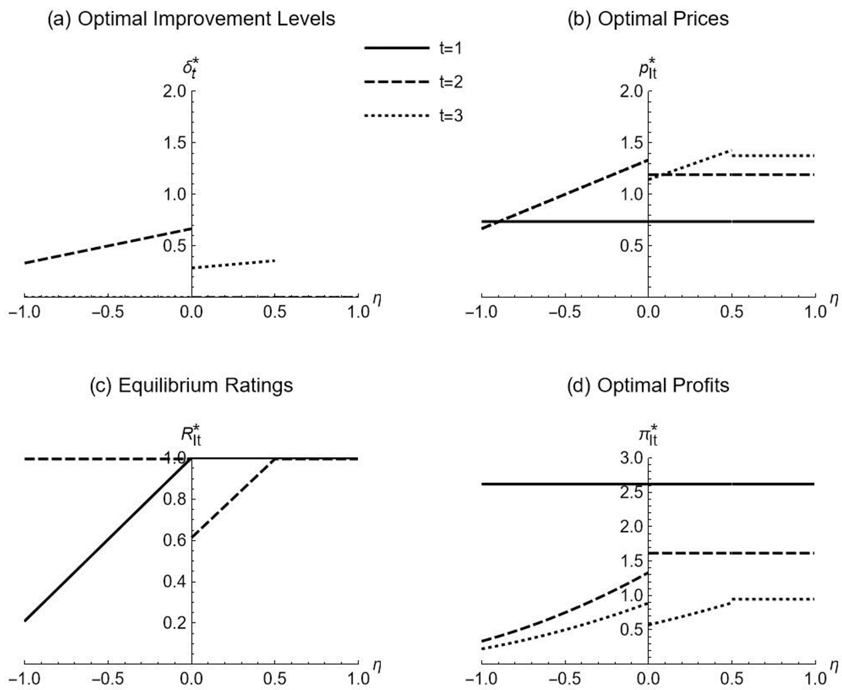

Figure 4 Optimal Improvements, Prices, Ratings, and Profits for High c Values ($V=2$, $c=1$)

results in consumers' lower willingness to pay, which drives lower optimal prices (over time). As we will discuss next, this price drop intensifies with a further increase in the improvement cost.

5.2. Results Based on Improvement Cost for a Given Product Performance

We next present the results and insights related to the progression of product improvements, prices, profits, and ratings with respect to the cost of improvement, c, for a given performance level, η. We first focus on examining the firm's optimal decision on product improvement, δ. Figure 5 illustrates that, if the firm (optimally) decides to improve its product when the cost is low, it aims for the highest possible performance. We observe in both panels of the figure that, in the low range of c, the firm fills the gap between the realized performance (-0.5 on the left and 0.25 on the right) and the maximum achievable level of 1. Conceivably, beyond a certain cost threshold, the firm decreases its improvement level with an increase in cost. As we also explained via figures 3–a and 4–a, depending on whether the average ratings are informative with respect to the true product performance, the firm improves its product in either the second or the third period, but not both.

Figure 6 illustrates the optimal prices against the improvement cost c, for a given η. At the outset of the time horizon, the firm has no information on the true product performance and makes the pricing decision merely based on the expected performance, $\mathbb{E}[H]$. Thus, we see a repeating pattern

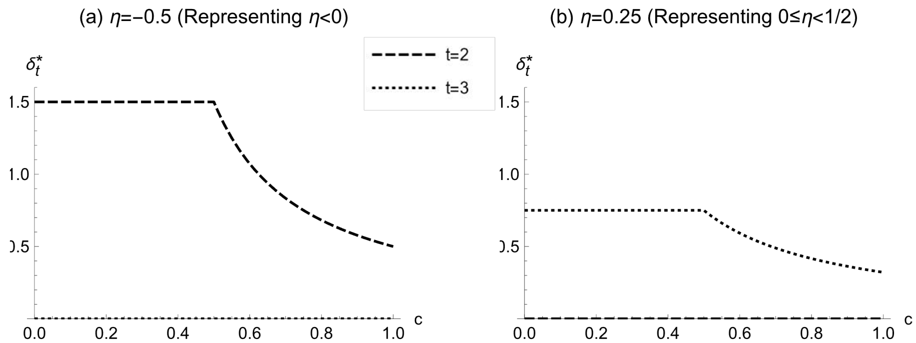

Figure 5 Optimal Product Improvement Levels for a Given η **(**$V=2$**)**

for p^*_{I1} across the panels, i.e., the first-period price is independent of η. One would expect that, with an increase in the improvement cost, the firm (at least partially) passes the cost on to its customers, and thus the optimal price rises. We have such an observation when c is relatively high. On the contrary, the optimal price interestingly decreases as the improvement cost increases in the low-cost range—wherein the firm optimally targets the maximum performance level (in either the second or the third period) if it improves the product at all. To understand the decreasing (first-period) price in this range, consider the customers' performance expectations in (16) and (17), which are (weakly) decreasing in the first-period price, p_1. The intuition is that, given the future consumers' belief about the first-period price ($\hat{p}_1$), a price decrease helps boost the average ratings, resulting in a superior performance perception in turn. As product improvement becomes more expensive (in the low-c range), the firm decreases the price with the motivation of narrowing the gap between the performance perception and the maximum possible performance of 1. Put another way, given $\hat{p}_1$, the firm has the incentive to decrease p_1 in an attempt to lower the future improvement cost. In rational expectation equilibrium, consumers can anticipate such a pricing strategy and match their price expectations with the firm's price decision, i.e., $\hat{p}_1 = p_1$ in equilibrium.

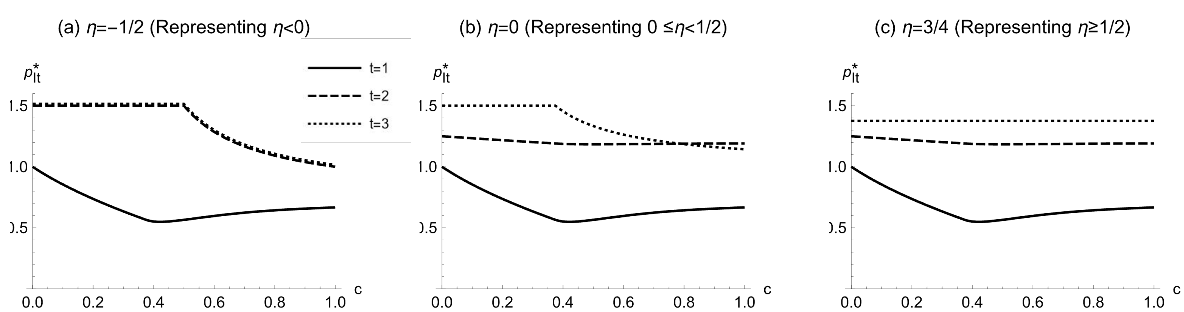

Figure 6 Optimal Prices for a Given η **(**$V=2$**)**

We now focus on the optimal prices in the second and third periods. Figure 6–a shows that both of these prices are initially constant and then decreasing in c when $\eta < 0$. The third-period price follows a similar pattern when $0 \leq \eta < 1/2$ as shown in Figure 6–b. In these ranges, for a sufficiently low improvement cost, the firm applies a maximum product improvement (as we discussed earlier), resulting in the highest level of consumer willingness to pay. Thus, in the low-cost range, the optimal prices are at the highest level and independent of cost. As the improvement cost increases beyond a threshold, the firm decreases the product improvement magnitude, reducing the consumers' willingness to pay and thus the optimal second-period price for $\eta < 0$ and the optimal third-period price for $\eta < 1/2$. Figures 6–b and 6–c also show that, when $\eta \geq 0$, the second-period price generally follows the first-period price pattern, that is, first decreasing and then increasing—although on a compressed scale. In this η range, the ratings from the first period do not convey any information on the true product performance, but are only enough to rule out the negative η range. Therefore, the intuition that explains the pattern of the first-period price facing no information applies to the second-period price when $\eta > 0$. Lastly, Figure 6–c shows that the third-period price is independent of the cost when $\eta \geq 1/2$. In this case, the true product performance is never revealed by the ratings from prior periods and no product improvement happens in any of the periods. Hence, the improvement cost becomes irrelevant to the firm's pricing decision.

Figure 7 demonstrates the optimal forward-looking expected profits against the improvement cost (c) for a given true product performance (η). First, consider a case wherein the true product performance is poor (i.e., η is close to -1) as shown in Figure 7–a. Recall that, when the improvement cost is cheap, the firm improves the product performance to its maximum level in the second period. Consequently, the second- and third-period optimal prices become equal. Since the firm incurs the improvement cost in the second period, as long as the firm's profit is positive in the second period, so is its second-period profit-to-go (because the third-period profit equals the second-period profit plus the improvement cost). As the improvement cost increases moderately, achieving the maximum performance requires a considerable investment due to the poor baseline product performance. Interestingly, in this case, the firm tolerates a *transitory loss* in the second period in the hope of achieving the maximum performance (and thus the maximum willingness to pay) in the last period. As a result of this short-term loss, the firm's second-period profit-to-go is less than the firm's third-period profit in the discussed (moderate) range of improvement cost as evident in Figure 7–a. Other trends shown in Figure 7 are intuitive and consistent with the earlier explanations: Profit functions are (weakly) decreasing in the cost of improvement and forward-looking profits are decreasing over time (unless the firm optimally undergoes a transitory profit loss, as just explained).

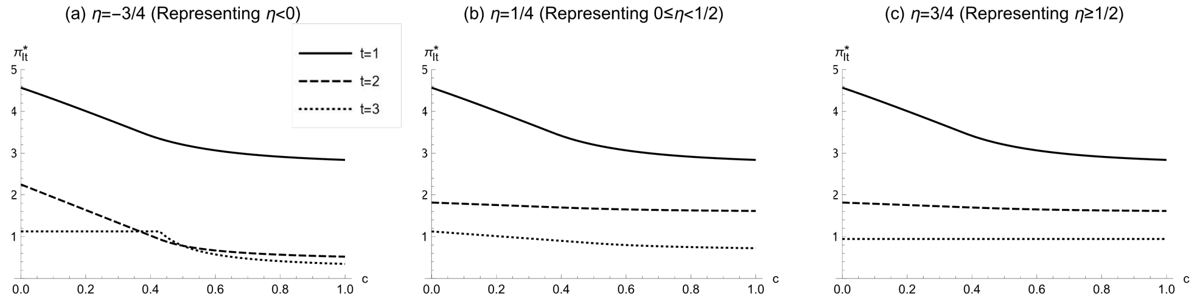

Figure 7 Optimal Forward-Looking Expected Profits for a Given η ($V = 2$)

The result we just explained provides an example of dynamics that cannot be captured via a two-period model. We can observe in 7–a that the profit-to-go trend between the second and third periods does not mimic the trend between the first two periods. Considering the intuition behind the profit-to-go decrease (over time), we argue that a two-period model does not yield a similar result. The reason is that, based on the explained intuition, a profit-to-go drop in the second period should be prompted by a maximum performance improvement in the first period, which cannot happen in the absence of consumption history (and thus product reviews). On the other hand, one can see that the results from the current setting would extend to models with more than three periods. To clarify that point, note that the information gleaned by product reviews in each period sheds light on the true product performance only if it is less than initially expected, leaving the remaining region uninformed. To be specific, the insights we obtain for the entire η range in the first period apply to $\eta \in [0, 1]$ in the second period and to $\eta \in [1/2, 1]$ in the third period. One would reasonably expect the trend to extend beyond three periods should we need to consider a general multi-period setting—thus justifying the parsimony of our three-period model.

Finally, Figure 8 illustrates the customer ratings against c for a given η. In case the product performance turns out to be poorer than expected (i.e., $\eta < 0$) at the end of the first period, the average customer rating exhibits a non-monotonic behavior, first increasing and then decreasing in c. As we see in (2), the average rating (if imperfect) is only sensitive to, and a decreasing function of, the price for a given realized performance. As a result, we can trace the pattern of the first-period average rating (when $\eta < 0$) back to the progression of the optimal first-period price in Figure 6. For $0 \leq \eta < 1/2$ (see Figure 8–b), the second-period ratings follow the pattern of the first-period ratings for $\eta < 0$—though on a compressed scale. As we described earlier, the perfect ratings under both $\eta < 0$ and $\eta \geq 0$ are due to the coincidence of the true product performance and customers' inference of the performance in equilibrium. In the following section, we elaborate on the differences between the two discussed scenarios and provide further managerial insights.

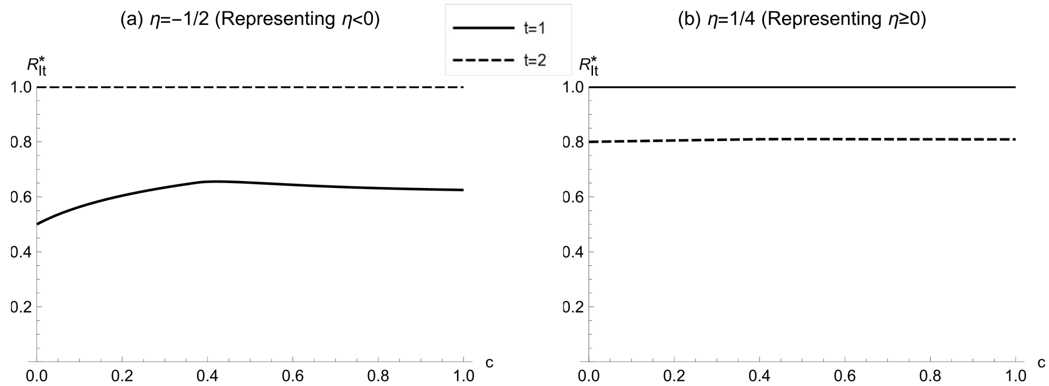

Figure 8 Equilibrium Ratings for a Given η ($V = 2$)

6. Impact of Product Improvement

We next compare the optimal prices, profits, and customer ratings between the two scenarios without and with the possibility of product improvement, respectively discussed in §4 and §5. Proposition 3 formalizes the price comparisons between the two scenarios.

Proposition 3 *The optimal prices in the scenarios with and without the possibility of product improvement compare as follows.*

(i) $p^*_{I1} > p^*_{N1}$ *when* $V \le 9$ *and* $c < \frac{3V-2}{10V}$ *or* $V > 9$ *and* $c < \bar{c}_1$ *where* $\bar{c}_1$ *is the unique solution to the following equation within* $(\frac{1+V}{4V}, \frac{1+V}{2V})$.

$$8V^3c^3 - 12V^2(V+1)c^2 + 2V(3V^2+9V+1)c - V^3 - 3V^2 - 3V - 1 = 0 \tag{21}$$

(ii) $p^*_{I2} > p^*_{N2}$ *for any* V *and* c *if* $\eta < 0$, *and for* $V \le 4$ *and* $c < \frac{6V+1}{20V}$ *or* $V > 4$ *and* $c < \bar{c}_2$ *if* $\eta \ge 0$. $\bar{c}_2$ *is the unique solution to the following equation within* $(\frac{1+V}{4V}, \frac{1+V}{2V})$.

$$16V^3c^3 - 24V^2(V+1)c^2 + V(12V^2+30V+13)c - 2V^3 - 6V^2 - 6V - 2 = 0 \tag{22}$$

(iii) *For any* V *and* c, $p^*_{I3} > p^*_{N3}$ *if* $\eta < 1/2$, *and* $p^*_{I3} = p^*_{N3}$ *otherwise.*

Proposition 3 shows that the product improvement capability results in a higher (lower) optimal first-period price if product improvement is (not) sufficiently cheap. A similar result applies to the second-period price when the realized performance in the first period meets or exceeds the a priori expectation. Otherwise, the scenario with improvement always yields a higher optimal second-period price than the scenario without improvement. The third-period price in the scenario with improvement is *always* at least equal to that in the scenario without improvement. Figure 9 illustrates the price comparisons in Proposition 3.

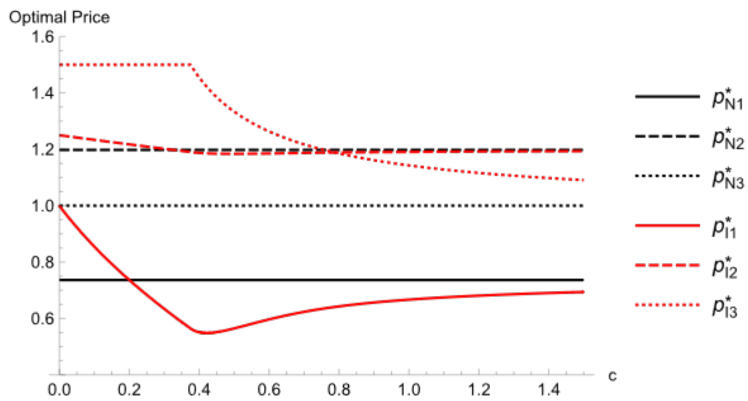

Figure 9 Comparison of Optimal Prices between Scenarios with and without Improvement ($\eta = 0$)

Focusing on the first-period price, we observe that the ability of product improvement results in a higher price when improvement is cheap and a lower price when the improvement cost is high enough. Interestingly, the firm adopts different pricing strategies in the two scenarios (with and without product improvement) even though it sells *identical* products (with the same performance levels) in the first period, before implementing any improvement. The intuition behind the price difference lies in the firm's motivation to signal future customers on the product improvement size, as we elaborated in §5.2. To reiterate, when the improvement cost is too low, future customers expect a maximum product performance (and rationally so) regardless of the product's performance status in the first period. With next-to-zero improvement costs, the firm can afford such maximal improvements without having to lower its price, a strategy it would adopt to stimulate high ratings and high status quo performance perceptions when improvement becomes slightly more expensive. As the improvement cost exceeds a threshold, the firm starts to increase its price to cover the higher costs of potential product improvements.

As we explained in the previous sections, when $\eta \geq 0$, the second-period prices mimic the pattern of the first-period prices (described above), and otherwise, exactly coincide with the third-period prices. As Figure 9 illustrates (for the representative case of $\eta = 0$), the product improvement capability leads to a higher optimal third-period price when $\eta < 1/2$ because, in this range, the firm has already improved the product in one of the previous periods and thus optimally charges a higher price for the superior performance (relative to the scenario with no improvement). When $\eta \geq 1/2$, due to no improvements in the first two periods, the third-period prices match between the two scenarios (with and without improvement). Figure 9 also illustrates the intuitive result that, as the improvement cost goes to infinity, even in the presence of improvement capability, the firm optimally withholds from product improvement (in the asymptotic case) and the prices in all three periods converge to those in the absence of product improvement.

Figure 10 provides a comparison of the forward-looking profits in equilibrium between the two scenarios with and without improvement. As one would expect, the scenario with improvement yields higher equilibrium profits because, in that scenario, the firm has the additional leverage of deciding on the improvement level besides adjusting its price strategy. In other words, with the improvement potential, the firm has the option not to utilize it and to replicate the optimal pricing strategy in the no-improvement scenario—generating a weakly inferior profit.

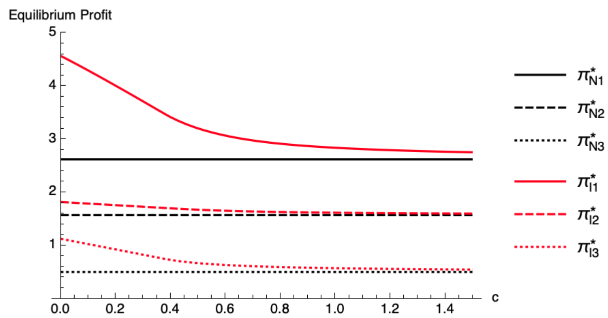

Figure 10 Comparison of Equilibrium Profits(-to-Go) between Scenarios with and without Improvement ($\eta = 0$)

Next, we compare via Proposition 4 the customer ratings in equilibrium under the two scenarios with and without improvement.

Proposition 4 *The equilibrium ratings in the scenarios with and without the possibility of product improvement compare as follows.*

(i) For $\eta < 0$, $R^*_{I1} < R^*_{N1}$ if $p^*_{I1} > p^*_{N1}$ and $R^*_{I1} > R^*_{N1}$ if $p^*_{I1} < p^*_{N1}$.
For $\eta \geq 0$, $R^*_{I1} = R^*_{N1}$.

(ii) For $\eta \in [0, \frac{1}{2})$, $R^*_{I2} < R^*_{N2}$ if $p^*_{I2} > p^*_{N2}$ and $R^*_{I2} > R^*_{N2}$ if $p^*_{I2} < p^*_{N2}$.
For $\eta < 0$ or $\eta \geq \frac{1}{2}$, $R^*_{I1} = R^*_{N1}$.

Proposition 4 establishes that, in each of the first two periods, the order of equilibrium ratings in the two scenarios (with and without improvement) is opposite to the order of optimal prices (in the η ranges where improvement happens). That is, whenever the optimal price in the improvement scenario is higher than that in the no-improvement scenario, the equilibrium rating in the improvement scenario is lower than that in the no-improvement scenario, and vice versa. Considering the price comparisons (provided in Proposition 3), the presented result implies that, when the improvement cost is cheap (expensive), the average rating in the period preceding an optimal product improvement is inferior (superior) to the average rating in the same period under the no-improvement scenario. This result suggests that customers rate the same product differently based on only the *prospect* of product improvement—which only the future buyers will enjoy. The difference in the average ratings corresponds to the (previously described) result that the firm prices the same product differently based on the prospect of improvement in the future, and the average ratings are inversely related to the prices. Figure 11–(a) illustrates the described progression of the average ratings in the two scenarios against the unit improvement cost (c)—with a focus on the second-period ratings and a correspondingly adjusted scale to help visibility. Additionally, our results show that, when the improvement cost goes to infinity, the ratings with and without improvement converge since product improvement (in the scenario where it is feasible) becomes a less viable strategy. Figure 11–(b) demonstrates the average ratings against the true product performance (η) for an improvement cost of zero. This figure confirms the finding that, when product improvement is affordable, customers reviews for the same product are more positive with the prospect of an upcoming improvement—which optimally occurs if the realized performance is less than expected, i.e., $\eta < 0$ in the first period and $0 \le \eta < 1/2$ in the second period. Otherwise, the average ratings under the two scenarios coincide.

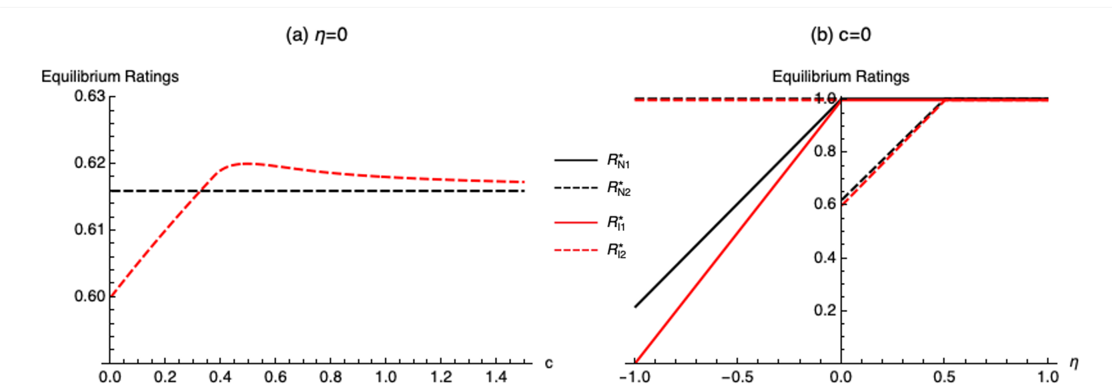

Figure 11 **Comparison of Equilibrium Ratings between Scenarios with and without Improvement**

7. Conclusion

Thanks to the growing accessibility of product review platforms, product ratings have increasingly become integral to consumers' purchase decisions. While customers typically view a newly released product as a wild card, they receive a signal of performance from prior customer reviews that potentially accompany a relatively established product. This signal can be imperfect due to the customers' limited information on (1) the price at which a (former) reviewer has purchased the product, and (2) the extent to which the product performance has exceeded or fallen short of the reviewer's ex ante expectation. Such uncertainty creates the opportunity for the seller to leverage pricing as a means of stimulating the reviews used by future customers when making their purchase decisions. Yet, the seller's capacity to sway later customers' decisions by stimulating earlier ratings is subject to the customers' "rational expectation" of this strategy. We have developed a multi-period game-theoretic model to shed light on such an interplay between the seller and consumers. As product ratings reveal the product performance (to both sides) over time, consumers narrow down their expectations of the product performance and make more informed purchase decisions; correspondingly, the firm adjusts its price and potentially product performance—via improvement.

Our first set of results show that, if the realized product performance exceeds the ex ante expectation in a period, the ensuing average rating is perfect, but does not precisely convey the true product performance to the firm and future consumers. Otherwise, consumption yields less-than-perfect average ratings, which future consumers can use to precisely infer the true product performance in equilibrium. Furthermore, our results establish that the firm should optimally increase its price when the product's realized performance is greater than expected, and decrease its price when the realized performance is *significantly* less than expected. Interestingly, the firm should increase its price when the realized performance is *moderately* less than expected. The intuition lies in the firm's lack of ability to signal high performance via price-driven ratings since customers can deduce the exact performance from product ratings in this case. When the firm possesses an improvement capability and optimally implements it, we find that the improvement size decreases in the true product performance if improvement is cheap because the firm aims for maximum performance. On the other hand, if the improvement cost is high, the optimal improvement size increases in the (baseline) true performance due to customers' higher willingness to pay, and thus higher optimal prices, which cover the costs of larger improvements. Finally, we show that the *potential* of product improvement per se changes the firm's optimal pricing strategy for the same product—i.e., resulting in a higher initial price when the cost is low and a lower initial price when the cost is high—even before implementing any improvement. That is due to the firm's use of pricing for the purpose of signaling the future improvement degree and stimulating purchases.

Our study presents managerial guidelines with respect to pricing and improvement of a product across its different generations, considering customer ratings which signal the product performance. Consistent with the examples provided earlier, including many appliances and software applications with long times between generations, we focus on products that customers buy the available version of whenever the need emerges. Correspondingly, the analysis of customers' strategic postponement of purchase (in expectation of a new version) is beyond the scope of this paper. Furthermore, we consider products that have the same core functionality across generations, thus customers do not consider replacing them in the short run. These limitations can guide the agenda for future research involving customer ratings, pricing, and product improvement.

Notes

[1]https://www.irobot.com/en_US/roomba.html

[2]https://itouchless.com/products/13-gallon-deodorizer-sensor-trash-can-dzt13p

[3]https://squareup.com/us/en/hardware/terminal

[4]https://www.microsoft.com/en-us/microsoft-365/p/office-home-student-2021/CFQ7TTC0H8N8

[5]https://amz.run/5nAl.

[6]https://support.microsoft.com/en-us/office/how-do-i-upgrade-office-ee68f6cf-422f-464a-82ec-385f65391350

[7]Note that the mentioned products are different from the subscription-based services provided by the same companies (e.g., Office 365). The subscription-based products receive updates and upgrades without imposing any charges on the subscribers. Such business models are not within the scope of our research.

[8]https://forums.steinberg.net/t/why-do-i-have-to-pay-for-cubase-10-5-if-i-already-pay-for-cubase-10-complete/138502

[9]For analytical tractability, we assume that $V \geq 1$, meaning that a consumer with the maximum idiosyncratic value will buy a product at zero price regardless of where the product-specific value component (H) lands within the $[-1, 1]$ support.

[10]For example, see https://www.irobot.com/en_US/comparison-chart.html, which provides a comparison chart for the features of different models of vacuums the company has offered. A customer can use this table to learn about the product improvements in a model relative to an earlier one. Similar comparison charts for other products mentioned in the introduction are available on the company websites.

[11]These ratings follow the same pattern as in the scenario without product improvement. We further compare the customer ratings under the two scenarios with and without product improvement in §6.

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Online Appendix:
Proofs of Lemmas, Propositions and Corollaries

This supporting document presents the proofs of the propositions and corollaries.

Proof of Lemma 1 Consumers arriving in the first period purchase the product if their expected utility is non-negative, that is, $\mathbb{E}[U_1^i] = v^i + \mathbb{E}[H] - p_1 = v^i - p_1 \geq 0$. Since v^i follows a uniform distribution on the $[0, V]$ support, the size of the participating population in the first period is $V - p_1$. After consuming the product, a customer with $\eta \geq 0$ realizes a non-negative utility and therefore leaves a rating of 1. On the other hand, customers who realize a negative η receive positive consumption utility only if $v^i + \eta - p_1 \geq 0 \Rightarrow v^i \geq p_1 - \eta$. So, the size of the market with a positive utility is $V - p_1 + \eta$ if this expression is non-negative, and 0 otherwise. Therefore, the proportion of positive ratings relative to the population that initially bought the product is $R_1(\eta, p_1) = (\frac{V-p_1+\eta}{V-p_1})^+ = (1 + \frac{\eta}{V-p_1})^+$. For analytical tractability, we keep V large enough to avoid the scenario with a zero first-period average rating, simplifying the expression into that in (2). We will derive the lower bound of the condition on V later in the paper. Furthermore,

Observing $R_1(\eta, p_1)$ and assuming a price of $\hat{p}_1$ in the first period, the second-period consumers' conditional expectation of H is derived by inverting Equation (2) as below. For brevity, we slightly abuse the notation, referring to $R_1(\eta, p_1)$ as R_1 and to $R_2\big(\eta, p_2, R_1(\eta, p_1)\big)$ as R_2 at times.

$$\mathbb{E}[H|R_1, \hat{p}_1] = \begin{cases} 1/2 & R_1 = 1 \\ \max\{-1, -(1 - R_1)(V - \hat{p}_1)\} & R_1 < 1 \end{cases}$$

Plugging R_1 from (2) into the above expression, we obtain the second-period consumers' expectation of H conditional on η, p_1 and $\hat{p}_1$ as follows, which turns into the expression in (4) considering the fact that, in equilibrium, $p_1 = \hat{p}_1$ and $\eta \in [0, 1]$.

$$\hat{\eta}_{N2}(p_1) \equiv \mathbb{E}[H|\eta, p_1, \hat{p}_1] = \begin{cases} 1/2 & \eta \geq 0 \\ \max\{-1, \eta \frac{V-\hat{p}_1}{V-p_1}\} & n < 0 \end{cases}$$

Forming the product performance expectation based on their observation of the ratings from the first period, second-period consumers purchase the product at p_2 if $E[U_2^i] = v^i + \hat{\eta}_{N2}(p_1) - p_2 \geq 0 \Rightarrow v^i \geq p_2 - \hat{\eta}_{N2}(p_1)$. The first piece of $\hat{\eta}_{N2}(p_1)$ in the above equation corresponds to the case in which second-period consumers observe $R_1 = 1$, ruling out the negative portion of H's distribution support on $[-1, 1]$. In this case, the total (participating) market size is $V - p_2 + 1/2$. If the true product performance turns out to be at least as good as the expectation in this case (1/2), all participating consumers leave a positive rating. Otherwise, the size of the market gaining positive utility is obtained by $v^i + \eta - p_2 \geq 0 \Rightarrow v^i \geq p_2 - \eta$, and the proportion of positive ratings by $\frac{V-p_2+\eta}{V-p_2+1/2}$. In the second piece of $\hat{\eta}_{N2}(p_1)$, the initial distribution of H can be narrowed down (after consumption) to a single value which hinges upon η, p_1, and $\hat{p}_1$. In equilibrium, second-period consumers' belief about the first-period price $\hat{p}_1$ matches the true price p_1, and these consumers can deduce the true product performance η. Since, only those (second-period) consumers who have a positive assessment of the product make a purchase and, in equilibrium, this assessment matches the true value of the product performance, the second-period rating in this case is 1. Putting the two cases together, we will obtain Equation (3) in the proposition statement.

We now turn our attention to the third-period consumers' expectation of the product performance after observing the ratings from the first and second periods. The following expression is directly obtained from (2) and (3) in each of the possible scenarios on R_1 and R_2. The first piece reflects the scenario where the average ratings in both periods are equal to 1, so we can reduce the feasible range of η to $[1/2,1]$ yielding the expected value of 3/4. The conditions of the second piece imply that $\eta \in [0,1/2)$. In this region, consumers can directly infer η by solving $R_2 = \frac{V-\hat{p}_2+\eta}{V-\hat{p}_2+1/2}$ based on their belief about the second period price, $\hat{p}_2$. The third piece is derived in the similar fashion, this time by solving the second piece in (2) for η.

$$\mathbb{E}[H|R_1,R_2,\hat{p}_1,\hat{p}_2] = \begin{cases} 3/4 & R_1=1, R_2=1 \\ R_2(V-\hat{p}_2+1/2)-(V-\hat{p}_2) & R_1=1, R_2<1 \\ (R_1-1)(V-\hat{p}_1) & R_1<1, R_2=1 \end{cases}$$

After plugging R_1 and R_2 from (2) and (3) respectively into the above function, we obtain the firm's (and customers') expectation of the product performance in the third period given the actual and assumed prices and well as the actual product performance. The result is shown in (5). To be more accurate, we should rewrite the expressions to impose the feasible ranges of η, but we forgo that level of rigor in the interest of focusing on the equilibrium outcome where $p_2 = \hat{p}_2$, which guarantees that each piece falls within the appropriate range.

□

Proof of Proposition 1 We solve the firm's problem backwards, starting from the third-period maximization problem in (6), then considering the problems in (7) and (8), respectively. We obtain the following solution for the third-period problem directly from the first-order condition.

$$p^*_{N3}(p_1,p_2) = \frac{V+\hat{\eta}_{N3}(p_1,p_2)}{2} \tag{23}$$

$$\pi^*_{N3}(p_1,p_2) = \frac{\left(V+\hat{\eta}_{N3}(p_1,p_2)\right)^2}{4V} \tag{24}$$

Plugging the third-period optimal profit into (7), we will have:

$$\max_{p_2} \tilde{\pi}_{N2}(p_1,p_2) = p_2\left(\frac{V-p_2+\hat{\eta}_{N2}(p_1)}{V}\right) + \mathbb{E}_H\left[\frac{\left(V+\hat{\eta}_{N3}(p_1,p_2)\right)^2}{4V}|R_1\right]$$

The expected optimal profit in the second term of the above function is conditional on the customer ratings from the first period. Upon observing $R_1 = 1$, the firm rules out the possibility of a negative product performance and calculates the expectation by using the first two pieces of (5) over the corresponding ranges of η, i.e., $\eta \geq 1/2$ and $0 \leq \eta \leq 1/2$, respectively. On the other hand, an imperfect first-period rating ($R_1 < 1$) points to the profit given by the third piece of the function in (5) and is fully informative as to the value of η. Consequently, the above function can be written as follows.

$$\tilde{\pi}_{N2}(p_1,p_2) = \begin{cases} p_2\left(\frac{V-p_2+\frac{1}{2}}{V}\right) + \int_0^{\frac{1}{2}} \frac{1}{4V}\left(V+\frac{V-p_2+\eta}{V-p_2+\frac{1}{2}}(V-\hat{p}_2+\frac{1}{2})-(V-\hat{p}_2)\right)^2 d\eta + \int_{\frac{1}{2}}^1 \frac{1}{4V}\left(V+\frac{3}{4}\right)^2 d\eta & \eta \geq 0 \\ \frac{p_2}{V}\left(V-p_2+\eta\frac{V-\hat{p}_1}{V-p_1}\right) + \frac{1}{4V}\left(V+\eta\frac{V-\hat{p}_1}{V-p_1}\right)^2 & \eta < 0 \end{cases}$$

After analyzing the first and second order conditions and subsequently replacing $\hat{p}_2$ with p_2 (to reflect the rational-expectation equilibrium solution), we derive the firm's optimal price in the second period as below.

$$p^*_{N2}(p_1) = \begin{cases} \frac{9+18V-\sqrt{6}\sqrt{6V^2+9V+2}}{24} & \eta \geq 0 \\ \frac{\left(p_1V+\hat{p}_1\eta - V(V+\eta)\right)^2}{2V(V-p_1)^2} & \eta < 0 \end{cases} \tag{25}$$

The second derivatives of the two specifications for $\eta \geq 0$ and $\eta < 0$ equal $-\left(\frac{2}{V} + \frac{1}{2(1+2V-2p_2)^2}\right)$ and $-2/V$, respectively, so the solution in (25) maximizes the profit.

Focusing on the first period, we expand in the following the forward-looking profit function in (8) by plugging in the second-period optimal price characterized in (25).

$$\begin{aligned} \tilde{\pi}_{N1}(p_1) = \frac{p_1}{V}(V-p_1) &+ \int_{-1}^{0} \tilde{\pi}_{N2}\left(p_1, \frac{\left(p_1V+\hat{p}_1\eta - V(V+\eta)\right)^2}{2V(V-p_1)^2}\right) d\eta \\ &+ \int_{0}^{1} \tilde{\pi}_{N2}\left(p_1, \frac{9+18V-\sqrt{6}\sqrt{6V^2+9V+2}}{24}\right) d\eta \end{aligned}$$

Subsequently, we replace $\hat{p}_1$ with p_1 in the above expression's first and second derivatives as below.

$$\frac{d\tilde{\pi}_{N1}(p_1)}{dp_1}|_{\hat{p}_1=p_1} = \frac{6V^2+12p_1^2-18p_1V-3V+2}{6V(V-p_1)}$$

$$\frac{d^2\tilde{\pi}_{N1}(p_1)}{dp_1^2}|_{\hat{p}_1=p_1} = -\frac{V+2(V-p_1)^2-1}{V(V-p_1)^2}$$

The first-period optimal price given in the proposition statement is the only solution which satisfies both the first- and the second-order conditions. The equilibrium price in the second period is derived by substituting the obtained price for p_1 and $\hat{p}_1$ in (25). Finally, we derive the equilibrium price in the third period by plugging the obtained prices (for the first and second periods) in (23), where $\hat{\eta}_{N3}(p_1, p_2)$ is given by (5).

□

Proof of Lemma 2 The proof logic generally follows that for Lemma (1). Since the firm does not improve its product in the first period, the initial average rating (R_1) equals that in the prior scenario—with no improvement. As we argue in the paper, the firm has no basis for product improvement when $R_1 = 1$ (or equivalently, $\eta \geq 0$), leading to $\hat{\eta}_{I2}(p_1, \delta_2) = 1/2$, similar to the baseline scenario. On the other hand, when $R_1 < 1$, the firm implements an improvement of size δ_2, which consumers can observe by reviewing the product version description in the second period. Thus, we obtain the pre-purchase performance expectation in (16).

In the second period, every consumer with $v^i \geq p_2 - \hat{\eta}_{I2}(p_1, \delta_2)$ purchases the product. For $\eta \geq 0$, the results are not any different from those in Lemma (1) because no improvement is implemented. Otherwise, for $\eta < 0$, a consumer can still solve the true product performance η in equilibrium with the knowledge of δ_2. Therefore, R_2 for the scenario with product improvement is the same as that for the baseline scenario, presented in (3). With the average ratings having the same functions as in Lemma (1), a consumer will have a similar expectation of the product performance level in the third period, but will also account for the added utility due to the observed product utility.

□

Proof of Proposition 2 Starting from the last period, we analyze the firm's problem in (18) and obtain $p_3^*(p_1,p_2,\delta_2,\delta_3) = \big(V + \hat{\eta}_{I3}(p_1,p_2,\delta_2,\delta_3)\big)/2$ and $\pi_3^*(p_1,p_2,\delta_2,\delta_3) = \big(V + \hat{\eta}_{I3}(p_1,p_2,\delta_2,\delta_3)\big)^2/4V - c\delta_3^2$. Considering the three pieces of the expected performance function in (17), it is evident that, except for when $0 \le \eta < 1/2$, the expected performance is not a function of δ_3, and the optimal value of product improvement in the third period is zero in ranges $\eta < 0$ and $\eta \ge 1/2$. The first order condition for $\pi_3^*(p_1,p_2,\delta_2,\delta_3)$ (with respect to δ_3) within $0 \le \eta < 1/2$ yields

$$\delta_3^{int} = \frac{p_2(1-2V) - (1+2V)(V+\eta) - \hat{p}_2(1-2\eta)}{(1+2V-2p_2)(1-4cV)}, \tag{26}$$

and the second derivative is $1/2V - 2c$. Defining

$$\psi(p_2,\eta) \equiv \frac{V - p_2 + \eta}{V - p_2 + 1/2}(V - \hat{p}_2 + 1/2) - (V - \hat{p}_2), \tag{27}$$

which is the second piece of the expected performance function in (17) after dropping the last term, the expression in (26) can be rewritten as

$$\delta_3^{int} = \frac{\psi(p_2,\eta) + V}{4cV - 1}. \tag{28}$$

Since the product performance upper bound is 1, the upper bound of $\psi(p_2,\eta)$ is $1 - \delta_3$, and thus $\delta_3 \in \big[0, 1 - \psi(p_2,\eta)\big]$. When the third-period profit is convex (i.e., $4cV - 1 \le 0$), we do not obtain a positive interior solution for δ_3. Correspondingly, within the non-negative range of δ_3, the profit function is convex increasing and δ_3 achieves its optimal value at $1 - \psi(p_2,\eta)$. In equilibrium (when $\hat{p}_2 = p_2$), the third period profit at $\delta_3 = 1 - \psi(p_2,\eta)$ is $\Big(1 + V\big(2 + V - 4c(1-\eta)^2\big)\Big)/4V$, which is always positive when $0 \le \eta < 1/2$ and $4cV - 1 \le 0$.

Now we consider the case in which the third-period profit is concave, entailing $4cV - 1 > 0$. In this case, the specification in (28) is always positive. We verify that $\delta_3^{int} \ge 1 - \psi(p_2,\eta)$ when $\frac{1}{4V} < c \le \frac{1+V}{4V(1-\eta)}$, and $\delta_3^{int} \in \big(0, 1 - \psi(p_2,\eta)\big)$ when $c > \frac{1+V}{4V(1-\eta)}$. In the former range, the optimal improvement level is $1 - \psi(p_2,\eta)$, and in the latter range, $\delta_3^* = \delta_3^{int}$. It is easy to show that the profit is positive at these optimal levels when the corresponding conditions (on c) hold. In summary, the optimal improvement size in the third period, followed by the optimal price and profit as functions of it, is as follows.

$$\delta_3^* = \begin{cases} \frac{\psi(p_2,\eta)+V}{4cV-1} & c > \frac{1+V}{4V(1-\eta)} \wedge 0 \le \eta < \frac{1}{2} \\ 1 - \psi(p_2,\eta) & c \le \frac{1+V}{4V(1-\eta)} \wedge 0 \le \eta < \frac{1}{2} \\ 0 & \eta < 0 \vee \eta \ge 1/2 \end{cases} \tag{29}$$

$$p_3^*(p_1,p_2,\delta_2) = \frac{V + \hat{\eta}_{I3}(p_1,p_2,\delta_2,\delta_3^*)}{2} \tag{30}$$

$$\pi_3^*(p_1,p_2,\delta_2) = \frac{\big(V + \hat{\eta}_{I3}(p_1,p_2,\delta_2,\delta_3^*)\big)^2}{4V} - c{\delta_3^*}^2 \tag{31}$$

in addition, $c = 1/4V$ is the inflection point for the profit function versus improvement (at the optimal price).

Having characterized the solution of the third-period problem, we now consider the second-period problem in (19), which we study under the two cases of $\eta \ge 0$ and $\eta < 0$ due to the two-piece specification of $\hat{\eta}_{I2}(p_1,\delta_2)$

in (16). In the positive η range, the specification of δ_3^* changes at not only $\eta = 1/2$ but also potentially $c = \frac{1+V}{4V(1-\eta)}$ if solving this equation for η yields a value within $(0, 1/2)$. We derive this threshold as

$$\bar{\eta} \equiv 1 - \frac{1+V}{4cV}, \tag{32}$$

below which $\delta_3^* = \frac{\psi(p_2,\eta)+V}{4cV-1}$ and above which $\delta_3^* = 1 - \psi(p_2, \eta)$—when $\bar{\eta} \in (0, 1/2)$. Consequently, we analyze the $\eta \geq 0$ case by considering the magnitude of $\bar{\eta}$ relative to the $(0, 1/2)$ interval.

Second-Period Forward-Looking Profit When $\eta \geq 0$

We analyze the $\eta \geq 0$ case by considering the magnitude of $\bar{\eta}$ relative to the $(0, 1/2)$ interval. When $\bar{\eta} \leq 0$, or equivalently, $c \leq \frac{1+V}{4V}$, the second-period forward-looking profit in 19 is rewritten as follows—using the specification of $\hat{\eta}_{I3}(p_1, p_2, \delta_2, \delta_3^*)$ in (17) and δ_3^* in (29), and that $\hat{\eta}_{I2}(p_1, \delta_2) = 1/2$ when $\eta \geq 0$.

$$\tilde{\pi}_{I2}(p_1, p_2, \delta_2) = p_2\Big(\frac{V - p_2 + 1/2}{V}\Big) - c\delta_2^2 + \int_0^{1/2} \left(\frac{(1+V)^2}{4V} - c\big(1 - \psi(p_2, \eta)\big)^2\right) d\eta + \int_{1/2}^1 \frac{(V+3/4)^2}{4V} d\eta \tag{33}$$

We can observe that, in the above expression, no term other than $-c\delta_2^2$ is a function of δ_2, and thus the optimal performance level in this case is simply zero. Consequently, with a slight abuse of notation and for the sake of brevity, we drop the cost term from the expression in the remaining part of the analysis for this case and express the forward-looking profit function as $\tilde{\pi}_{I2}(p_1, p_2)$. In case $\bar{\eta} \in (0, 1/2)$, or equivalently, $\frac{1+V}{4V} < c < \frac{1+V}{2V}$, the second-period forward-looking profit is expanded as below.

$$\begin{aligned}\tilde{\pi}_{I2}(p_1, p_2) = p_2\Big(\frac{V - p_2 + 1/2}{V}\Big) &+ \int_0^{\bar{\eta}} \left(\frac{\big(V + \psi(p_2, \eta) + \frac{\psi(p_2,\eta)+V}{4cV-1}\big)^2}{4V} - c\Big(\frac{\psi(p_2, \eta) + V}{4cV - 1}\Big)^2\right) d\eta \\ &+ \int_{\bar{\eta}}^{1/2} \left(\frac{(1+V)^2}{4V} - c\big(1 - \psi(p_2, \eta)\big)^2\right) d\eta + \int_{1/2}^1 \frac{(V+3/4)^2}{4V} d\eta\end{aligned} \tag{34}$$

Finally, when $\bar{\eta} \geq 1/2$ or $c \geq \frac{1+V}{2V}$, we have the following.

$$\begin{aligned}\tilde{\pi}_{I2}(p_1, p_2) = p_2\Big(\frac{V - p_2 + 1/2}{V}\Big) &+ \int_0^{1/2} \left(\frac{\big(V + \psi(p_2, \eta) + \frac{\psi(p_2,\eta)+V}{4cV-1}\big)^2}{4V} - c\Big(\frac{\psi(p_2, \eta) + V}{4cV - 1}\Big)^2\right) d\eta \\ &+ \int_{1/2}^1 \frac{(V+3/4)^2}{4V} d\eta\end{aligned} \tag{35}$$

We first analyze the maximization of the provided profit functions with respect to price by considering the first- and second-order conditions. After obtaining the interior solution of each of the functions, we verify that it yields a positive profit and is within $(0, V + 1/2)$, where $V + 1/2$ is the maximum price consumers are willing to pay in the second period—because, when $\eta \geq 0$, the expected product performance is 1/2. Eventually, we replace $\hat{p}_2$ with p_2 to impose the equilibrium criterion. The result is as follows.

$$p_2^* = \begin{cases} \frac{1}{24}\left(9 + 18V - \sqrt{9 + 12V(3 + 5c + 3V)}\right) & c \leq \frac{1+V}{4V} \\ \frac{3V}{4} + \frac{1}{24}\left(9 - \sqrt{\frac{6 - 3V\left(16c^3V^2 + 3c(4V+3) + 8c^2V(6V^2+6V-1) + 2(V^2+3V+3)\right)}{cV(1-4cV)}}\right) & \frac{1+V}{4V} < c < \frac{1+V}{2V} \\ \frac{1}{24}\left(9 + 18V - \sqrt{\frac{9(4V^2+4V+1) - 24cV(6V^2+9V+2)}{1-4cV}}\right) & c \geq \frac{1+V}{2V} \end{cases} \tag{36}$$

Second-Period Forward-Looking Profit When $\eta < 0$

In this case, we use the specifications in (16), (17), (29), and 31) to rewrite the second-period forward-looking profit function in (19) as below.

$$\tilde{\pi}_{I2}(p_1,p_2,\delta_2) = p_2\Big(\frac{V-p_2+\eta\frac{V-\hat{p}_1}{V-p_1}+\delta_2}{V}\Big) - c\delta_2^2 + \frac{\big(V+\eta\frac{V-\hat{p}_1}{V-p_1}+\delta_2\big)^2}{4V} \tag{37}$$

Note that, in this case, the customer narrows down the distribution of η to a single value (in equilibrium). Imposing the conditions that the market size in the first term of the above expression is non-negative and the product improvement in the third term ($\hat{\eta}_{I3} = \eta\frac{V-\hat{p}_1}{V-p_1} + \delta_2$) is at most equal to the upper-support of the product performance, we obtain the constraints (on p_2 and δ_2) for the following forward-looking profit maximization problem in the second period.

$$\begin{aligned} &\max_{p_2,\delta_2} \tilde{\pi}_{I2}(p_1,p_2,\delta_2) \\ &\text{s.t.} \quad p_2 \le V + \eta\frac{V-\hat{p}_1}{V-p_1} + \delta_2 \\ &\qquad\ \ \delta_2 \le 1 - \frac{V-\hat{p}_1}{V-p_1} \end{aligned} \tag{38}$$

We solve this problem by setting up the Lagrangian function as below.

$$\mathcal{L}(p_2,\delta_2,\lambda_1,\lambda_2) = \tilde{\pi}_{I2}(p_1,p_2,\delta_2) + \lambda_1\left(V+\eta\frac{V-\hat{p}_1}{V-p_1}+\delta_2-p_2\right) + \lambda_2\left(1-\frac{V-\hat{p}_1}{V-p_1}-\delta_2\right) \tag{39}$$

We then build the cases considering that $\frac{\partial\mathcal{L}}{\partial p_2} \le 0$ with $p_2 \ge 0$, $\frac{\partial\mathcal{L}}{\partial \delta_2} \le 0$ with $\delta_2 \ge 0$, and $\frac{\partial\mathcal{L}}{\partial \lambda_j} \ge 0$ with $\lambda_j \ge 0$, where $j \in \{1,2\}$, imposing the complementary slackness conditions. Leveraging the assumption that $V \ge 1 \ge \eta$, we obtain the solutions under the two cases with $\lambda_1 = 0, \lambda_2 > 0, p_2 > 0, \delta_2 > 0$ and $\lambda_1 = \lambda_2 = 0, p_2 > 0, \delta_2 > 0$, leading to the following solution.

$$(p_2^*,\delta_2^*) = \begin{cases} \left(\frac{1+V}{2}, 1-\eta\frac{V-\hat{p}_1}{V-p_1}\right) & c < \frac{1+V}{2V(1-\eta)} \\ \left(cV\frac{V(V+\eta)-(p_1V+\hat{p}_1\eta)}{(p_1-V)(1-2cV)}, \frac{V(V+\eta)-(p_1V+\hat{p}_1\eta)}{(p_1-V)(1-2cV)}\right) & c \ge \frac{1+V}{2V(1-\eta)} \end{cases} \tag{40}$$

First-Period Forward-Looking Profit

We analyze the first-period forward-looking profit in the three ranges bordered by the c thresholds in (36), namely, $\frac{1+V}{4V}$ and $\frac{1+V}{2V}$, which determine which piece is relevant when $\eta \ge 0$. For $\eta < 0$, the optimal solution we should select between the two pieces in (40) depends on the magnitude of c relative to $\frac{1+V}{2V(1-\eta)}$, which is always in $[\frac{1+V}{4V}, \frac{1+V}{2V})$ when $\eta \in [-1,0)$. Since we need to integrate the first-period forward-looking profit over $\eta \in [-1,1]$, it helps to solve $c = \frac{1+V}{2V(1-\eta)}$ for η and express the threshold in (40) as a function of this solution, which is given in the following.

$$\tilde{\eta} \equiv 1 - \frac{1+V}{2cV} \tag{41}$$

Note that $c < \frac{1+V}{2V(1-\eta)}$ translates into $\eta \ge \tilde{\eta}$, and the opposite case is equivalent to $\eta < \tilde{\eta}$. Moreover, we can easily verify that $\tilde{\eta} < 0$ as long as $c < \frac{1+V}{2V}$. Putting all the explained properties together, we expand the first-period forward-looking profit in (20) under each of the following cases—incorporating the appropriate pieces from (36) and (40).

When $c \leq \frac{1+V}{4V}$, the first-period forward-looking profit function is expanded as below.

$$\tilde{\pi}_{I1}(p_1) = p_1\big(\frac{V-p_1}{V}\big) + \int_{-1}^{0} \tilde{\pi}_{I2}\big(p_1, \frac{1+V}{2}, 1-\eta\frac{V-\hat{p}_1}{V-p_1}\big)d\eta$$
$$+ \int_{0}^{1} \tilde{\pi}_{I2}\big(p_1, \frac{9+18V-\sqrt{9+12V(3+5c+3V)}}{24}, 0\big)d\eta$$

When $\frac{1+V}{4V} < c < \frac{1+V}{2V}$, the first-period forward-looking profit function is written as below.

$$\tilde{\pi}_{I1}(p_1) = p_1\big(\frac{V-p_1}{V}\big) + \int_{-1}^{\tilde{\eta}} \tilde{\pi}_{I2}\big(p_1, cV\frac{V(V+\eta)-(p_1V+\hat{p}_1\eta)}{(p_1-V)(1-2cV)}, \frac{V(V+\eta)-(p_1V+\hat{p}_1\eta)}{(p_1-V)(1-2cV)}\big)d\eta$$
$$+ \int_{\tilde{\eta}}^{0} \tilde{\pi}_{I2}\big(p_1, \frac{1+V}{2}, 1-\eta\frac{V-\hat{p}_1}{V-p_1}\big)d\eta$$
$$+ \int_{0}^{1} \tilde{\pi}_{I2}\Big(p_1, \frac{3V}{4} + \frac{1}{24}\big(9 - \sqrt{\frac{6-3V\big(16c^3V^2+3c(4V+3)+8c^2V(6V^2+6V-1)+2(V^2+3V+3)\big)}{cV(1-4cV)}}\big), 0\Big)d\eta$$

Finally, when $c \geq \frac{1+V}{2V}$, we have the following.

$$\tilde{\pi}_{I1}(p_1) = p_1\big(\frac{V-p_1}{V}\big) + \int_{-1}^{0} \tilde{\pi}_{I2}\big(p_1, cV\frac{V(V+\eta)-(p_1V+\hat{p}_1\eta)}{(p_1-V)(1-2cV)}, \frac{V(V+\eta)-(p_1V+\hat{p}_1\eta)}{(p_1-V)(1-2cV)}\big)d\eta$$
$$+ \int_{0}^{1} \tilde{\pi}_{I2}\Big(p_1, \frac{1}{24}\big(9+18V-\sqrt{\frac{9(4V^2+4V+1)-24cV(6V^2+9V+2)}{1-4cV}}\big), 0\Big)d\eta$$

We begin analyzing the forward-looking profit function in the first period in the $c \leq \frac{1+V}{4V}$ range. In equilibrium, we derive the interior solution to the profit function as $p_1^* = \frac{3V}{4} - \frac{\sqrt{3V(40c+3V)}}{12}$ and establish that the second derivative of the profit function, $-\frac{4c}{(V-p_1)^2} - \frac{2}{V}$, is always negative. We also verify that the obtained price solution is always positive for $V \geq 1$ and cannot exceed V within the provided cost range. Moving to the second-period, for $c \leq \frac{1+V}{4V}$, the optimal second-period price and improvement level in equilibrium are given by the first pieces of (36) and (40), respectively, as provided below.

$$(p_2^*, \delta_2^*) = \begin{cases} \Big(\frac{1}{24}\big(9+18V-\sqrt{9+12V(3+5c+3V)}\big), 0\Big) & c \leq \frac{1+V}{4V} \wedge \eta \geq 0 \\ \big(\frac{1+V}{2}, 1-\eta\big) & c \leq \frac{1+V}{4V} \wedge \eta < 0 \end{cases} \tag{42}$$

Using the above solution, in the third period, we have that $\delta_3^* = 1-\eta$ if $\eta \in [0, \frac{1}{2})$, and $\delta_3^* = 0$ otherwise. Plugging the derived values into (17), we obtain $\eta_{I3}^* = \frac{3}{4}$ if $\eta \geq \frac{1}{2}$, and $\eta_{I3}^* = 1$ otherwise, subsequently yielding $p_3^* = \frac{V}{2} + \frac{3}{8}$ if $\eta \geq \frac{1}{2}$, and $p_3^* = \frac{1+V}{2}$ otherwise.

Now we analyze the profit function pertaining to $\frac{1+V}{4V} < c < \frac{1+V}{2V}$, which corresponds to the following interior solution.

$$p_1^* = \frac{1}{12V}\left(9V^2 - \sqrt{\frac{V}{c}\Big(6+3V\big(6-16c^3V^2+2V(3+V)-3c(2+V)(2+3V)+c^2(40V-6V^3)\big)\Big)}\right) \tag{43}$$

We verify the second-order condition for the optimality of the price and that, in the provided cost range, the obtained price is within $(0, V)$. Pooling the appropriate pieces from (36) and (40) and plugging in the optimal first-period price whenever necessary, we have the following second-period equilibrium price and improvement level for $c \in (\frac{1+V}{4V}, \frac{1+V}{2V})$.

$$(p_2^*, \delta_2^*) = \begin{cases} \big(\frac{3V}{4} + \frac{9}{24} - \sqrt{\frac{6-3V\big(16c^3V^2+3c(4V+3)+8c^2V(6V^2+6V-1)+2(V^2+3V+3)\big)}{24^2cV(1-4cV)}}, 0\big) & \eta \geq 0 \\ \big(\frac{1+V}{2}, 1-\eta\big) & 1-\frac{1+V}{2cV} \leq \eta < 0 \\ \big(\frac{cV(V+\eta)}{2cV-1}, \frac{V+\eta}{2cV-1}\big) & \eta < 1-\frac{1+V}{2cV} \end{cases} \tag{44}$$

As for the previous cost range, we first obtain the third-period improvement level and subsequently the expected performance and price in equilibrium as below.

$$\delta_3^* = \begin{cases} 1-\eta & 1-\frac{1+V}{4cV} \le \eta < 1/2 \\ \frac{V+\eta}{4cV-1} & 0 \le \eta < 1-\frac{1+V}{4cV} \\ 0 & \eta < 0 \vee \eta \ge 1/2 \end{cases} \Longrightarrow \hat{\eta}_{I3}^* = \begin{cases} 3/4 & \eta \ge 1/2 \\ 1 & \left(1-\frac{1+V}{4cV} \le \eta < 1/2\right) \vee \left(1-\frac{1+V}{2cV} \le \eta < 0\right) \\ \frac{V+4cV\eta}{4cV-1} & 0 \le \eta < 1-\frac{1+V}{4cV} \\ \frac{V+2cV\eta}{2cV-1} & \eta < 1+\frac{1+V}{2cV} \end{cases}$$

$$p_3^* = \begin{cases} \frac{V}{2}+\frac{3}{8} & \eta \ge 1/2 \\ \frac{1+V}{2} & \left(1-\frac{1+V}{4cV} \le \eta < 1/2\right) \vee \left(1-\frac{1+V}{2cV} \le \eta < 0\right) \\ \frac{2cV(V+\eta)}{4cV-1} & 0 \le \eta < 1-\frac{1+V}{4cV} \\ \frac{cV(V+\eta)}{2cV-1} & \eta < 1-\frac{1+V}{2cV} \end{cases}$$

Finally, we focus on the $c \ge \frac{1+V}{2V}$ range. We obtain the interior solution of the provided profit function in this range as below and verify that it is always within $(0,V)$ and that the second derivative of the profit function is negative in this range.

$$p_1^* = \frac{1}{12}\left(9V + \sqrt{\frac{3V\left(2c(3V^2+12V-8)-3V\right)}{2cV-1}}\right) \tag{45}$$

Using the obtained optimal first-period price, we derive the equilibrium levels in the second period as below.

$$(p_2^*, \delta_2^*) = \begin{cases} \left(\frac{1}{24}\left(9+18V-\sqrt{\frac{24cV(6V^2+9V+2)-36V(V+1)-9}{4cV-1}}\right), 0\right) & \eta \ge 0 \\ \left(\frac{cV(V+\eta)}{2cV-1}, \frac{V+\eta}{2cV-1}\right) & \eta < 0 \end{cases} \tag{46}$$

Moving onto the third period, we calculate the equilibrium improvement level, expected performance, and price as below.

$$\delta_3^* = \begin{cases} \frac{V+\eta}{4cV-1} & 0 \le \eta < 1/2 \\ 0 & (\eta \ge 1/2) \vee (\eta < 0) \end{cases} \Longrightarrow \hat{\eta}_{I3}^* = \begin{cases} 3/4 & \eta \ge 1/2 \\ \frac{V+4cV\eta}{4cV-1} & 0 \le \eta < 1/2 \\ \frac{V+2cV\eta}{2cV-1} & \eta < 0 \end{cases} \Longrightarrow p_3^* = \begin{cases} \frac{V}{2}+\frac{3}{8} & \eta \ge 1/2 \\ \frac{2cV(V+\eta)}{4cV-1} & 0 \le \eta < 1/2 \\ \frac{cV(V+\eta)}{2cV-1} & \eta < 0 \end{cases}$$

□

Proof of Proposition 3 We provide the proofs for the price comparisons in the same order as presented in the proposition statement.

(i) Considering $c \le \frac{1+V}{4V}$, we should compare p_{N1}^* in (9) and p_{I1}^* from part (i) of Proposition 2. After some algebraic simplification, we have that $p_{I1}^* > p_{N1}^*$ when $V \le 9$ and $c < \frac{3V-2}{10V}$ or $V > 9$ for any c within the feasible range in question. Next, we turn out attention to $\frac{1+V}{4V} < c < \frac{1+V}{2V}$. We first verify that $p_{I1}^* - p_{N1}^*$, where p_{I1}^* is obtained from part (ii) of Proposition 2, is positive at $c = \frac{1+V}{4V}$ and negative at $c = \frac{1+V}{2V}$. If we show there is a unique solution to $p_{I1}^* = p_{N1}^*$ within $c \in (\frac{1+V}{4V}, \frac{1+V}{2V})$, we prove that $p_{I1}^* - p_{N1}^*$ is positive when c is less than that solution. To that end, we set $p_{I1}^* = p_{N1}^*$, which yields the equation in (21). Let us denote the left-hand side of that equation by $g_1(c)$ as below.

$$g_1(c) = 8V^3c^3 - 12V^2(V+1)c^2 + 2V(3V^2+9V+1)c - V^3 - 3V^2 - 3V - 1$$

We have that $g_1(\frac{1+V}{4V}) = -\frac{1}{8}(V+1)(V-1)(V-9)$, which is negative when $V > 9$, and that $g_1(\frac{1+V}{2V}) = 3V^2+V-2$, which is positive when $V > 9$. Furthermore,

$$g_1''(c) = 24V^2\big(V(2c-1)-1\big).$$

Since $g_1''(c) < 0$ when $c \in (\frac{1+V}{4V}, \frac{1+V}{2V})$, $g_1(c)$ has exactly one solution in the provided cost range. Finally, pulling the appropriate price expressions, we can easily show that $p_{I1}^* > p_{N1}^*$ does not hold for any $c \geq \frac{1+V}{2V}$. Putting these results together with an earlier finding that $p_{I1}^* > p_{N1}^*$ for $V > 9$ and $c \leq \frac{1+V}{4V}$, we obtain the result in part (i) of the proposition statement.

(ii) For each range of parameters η and c, we obtain the appropriate pieces of the optimal second-period price from Proposition 1 for the scenario without product improvement, and from Proposition 2 for the scenario with product improvement. For $\eta < 0$, we have that $p_{N2}^* = \frac{V+\eta}{2}$, and that p_{I2}^* is either $\frac{V+1}{2}$ or $\frac{cV(V+\eta)}{2cV-1}$ depending on the value of c. We can easily show that both pieces of p_{I2}^* dominate p_{N2}^* (in the negative η range). For $\eta \geq 0$, $p_{I2}^* = \frac{\left(9+18V-\sqrt{9+12V(3+5c+3V)}\right)}{24}$ and $p_{N2}^* = \frac{9+18V-\sqrt{6}\sqrt{6V^2+9V+2}}{24}$ when $c \leq \frac{1+V}{4V}$. Contrasting the two expressions, $p_{I2}^* > p_{N2}^*$ for $c < \frac{6V+1}{20V}$ and $V \leq 4$. Furthermore $p_{I2}^* > p_{N2}^*$ when $c > \frac{1+V}{2V}$ and $\eta \geq 0$ for any $V > 1$. Finally, when $c \in (\frac{1+V}{4V}, \frac{1+V}{2V})$, we use an approach similar to that in part (i) of the proof, as follows. Denote by $g_2(c)$ the left-hand side of the equation in part (ii) of the proposition statement.

$$g_2(c) = 16V^3c^3 - 24V^2(V+1)c^2 + V(12V^2+30V+13)c - 2V^3 - 6V^2 - 6V - 2$$

We have that $g_2(\frac{1+V}{4V}) = -\frac{1}{4}(V+1)V(V-4)$, which is negative when $V > 4$, and that $g_2(\frac{1+V}{2V}) = \frac{1}{2}(6V+1)(V+1)$, which is positive when $V > 4$. Furthermore, $g_2'(c) = V\Big(13+6V\big(2V(1-2c)^2-8c+5\big)\Big)$, which is always positive for $c \in (\frac{1+V}{4V}, \frac{1+V}{2V})$. Thus, $g_2(c)$ has a unique solution in the given c range. We derive the condition in part (ii) of the proposition since $p_{I2}^* - p_{N2}^*$ is positive at $c = \frac{1+V}{4V}$ and negative at $c = \frac{1+V}{2V}$.

(iii) The provided result is simply obtained by directly comparing the pieces of the optimal third-period prices between the two scenarios with and without product improvement, for $\eta \geq 1/2$ and $\eta < 1/2$.

□

Proof of Proposition 4 From Lemma 1, we have the following.

$$R_1(\eta,p_1) = \begin{cases} 1 & \eta \geq 0 \\ 1+\frac{\eta}{V-p_1} & \eta < 0 \end{cases} \Longrightarrow \frac{\partial R_1(\eta,p_1)}{\partial p_1} = \begin{cases} 0 & \eta \geq 0 \\ \frac{\eta}{(V-p_1)^2} & \eta < 0 \end{cases}$$

$$R_2\big(\eta,p_2,R_1(\eta,p_1)\big) = \begin{cases} 1 & R_1(\eta,p_1)=1, \eta \geq \frac{1}{2} \\ \frac{V-p_2+\eta}{V-p_2+1/2} & R_1(\eta,p_1)=1, 0 \leq \eta < \frac{1}{2} \\ 1 & R_1(\eta,p_1) < 1, \eta < 0 \end{cases}$$

$$\Longrightarrow \frac{\partial R_2\big(\eta,p_2,R_1(\eta,p_1)\big)}{\partial p_2} = \begin{cases} 0 & R_1(\eta,p_1)=1, \eta \geq \frac{1}{2} \\ \frac{4\eta-2}{(2V-2p_2+1)^2} & R_1(\eta,p_1)=1, 0 \leq \eta < \frac{1}{2} \\ 0 & R_1(\eta,p_1) < 1, \eta < 0 \end{cases}$$

We can see that $\frac{\eta}{(V-p_1)^2} < 0$ when $\eta < 0$ and that $\frac{4\eta-2}{(2V-2p_2+1)^2} < 0$ when $0 \leq \eta < \frac{1}{2}$. Therefore, in the provided ranges, the average ratings are inversely related to the prices, establishing the results in the proposition statement (for the corresponding ranges). Outside the discussed η ranges, the ratings equal 1, irrespective of the scenario.

$\square$